

LOCAL-GLOBAL PRINCIPLES FOR CURVES OVER SEMI-GLOBAL FIELDS

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ABSTRACT. We investigate local-global principles for Galois cohomology, in the context of function fields of curves over semi-global fields. This extends work of Kato's on the case of function fields of curves over global fields.

1. INTRODUCTION

A basic theme in field arithmetic is the passage from a field F to the field of rational functions in one variable $F(x)$. In this paper, we discuss how local-global principles behave with respect to such transitions. In particular, suppose we are given a collection of overfields $F_v, v \in \Omega$, of F . A natural question to ask is, when one has knowledge of local-global principles for F with respect to the overfields F_v , what local-global principles may be obtained for the field $F(x)$ with respect to the overfields $F_v(x)$. More generally, one may also replace $F(x)$ by the function field $F(Y)$ of an algebraic curve Y/F .

Such questions have been asked before, primarily in the case that F is a global field, and some results have been obtained for the third Galois cohomology group [Kat86], for Witt groups of quadratic forms [PS96], for the Brauer group [LPS14], and for torsors over simply connected, semisimple groups of classical type [PP03]. A theme that has emerged in these investigations is that as one adjoins a transcendental element to a field, one expects to find local-global principles one degree higher than previously.

The first main instance of this theme concerned the theorem of Albert-Brauer-Hasse-Noether, which states a local-global principle for the Brauer group of a global field F , or equivalently for $H^2(F, \mu_m)$. Such a principle need not hold for $F(Y)$, if Y is a curve over a global field F . But in [Kat86], a local-global principle was shown in this context for $H^3(F(Y), \mu_m^{\otimes 2})$, with respect to the places of F .

In recent years, local-global principles have been studied for semi-global fields; i.e., one-variable function fields F over a complete discretely valued field K . Such principles were shown in [CPS12] and [HHK15] for $H^1(F, G)$ in the context of a rational linear algebraic group G ; and in [HHK14] for $H^n(F, \mu_m^{\otimes n-1})$ with respect to the discrete valuations on F if $n \geq 2$. The above theme suggests that one should expect a local-global principle for $H^n(F(Y), \mu_m^{\otimes n-1})$ with respect to the discrete valuations on a semi-global field F for sufficiently large values of $n \leq \text{cd}(F(Y))$. We prove such a result in Theorem 5.2 for $n \geq 3$ in the case that Y is the projective line over F , and for more general curves Y/F in Theorem 5.5 for $n \geq 3$ if K is a local field.

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In several papers (see [CH15], [HSS15],[HS16]), local-global principles for semi-global fields F were considered with respect to a smaller class of discrete valuations; viz., those that are trivial on the underlying complete discretely valued field K . In that situation, a local-global principle holds less often, since the hypothesis is weaker. In Theorem 5.12 we consider local-global principles for $H^n(F(x), \mu_m^{\otimes n-1})$ with respect to this smaller set of valuations on a semi-global field F , showing that it fails for $n = 4$ but holds under certain hypotheses (e.g., good reduction) for $n = 3$.

This manuscript is organized as follows. We introduce the set-up and discuss some classical examples in Section 2, followed by preliminary results in Section 3. In Section 4, after establishing our notation, we discuss the relationship between two types of local-global principles on a function field $E = F(Y)$ over a semi-global field F : with respect to the discrete valuations on E , and with respect to those on F (following Kato). Our main results appear in Section 5, first with respect to all the divisorial discrete valuations on F (in Section 5.1) and then with respect to those that are trivial on the underlying complete discretely valued field K (in Section 5.2).

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2. DEFINITIONS AND BASIC EXAMPLES

Let F be a field and let $F_v, v \in \Omega$ be a collection of overfields $F_v \supseteq F$. If E/F is a finite extension we set $E_v = E \otimes_F F_v$ (which is not necessarily a field). On the other hand, if Y/F is a geometrically integral algebraic variety and $E = F(Y)$, let E_v be the function field of the variety Y_{F_v} . For μ a $\text{Gal}(\bar{E}/E)$ -module we set

$$(1) \quad \text{III}_{\Omega}^n(E, \mu) = \ker \left[H^n(E, \mu) \rightarrow \prod_{v \in \Omega} H^n(E_v, \mu) \right].$$

We are interested to determine this group, and, in particular, to determine whether or not this group is trivial for certain fields F and collections $F_v, v \in \Omega$ of overfields. Our guiding principle is to understand how this group should depend on the quantities $n, i = \text{cd } F$, and $d = \dim Y$. We set $d = 0$ if E is a finite extension of F . We will eventually be mostly interested in the case where F is a function field in one variable over a complete discretely valued field, with a set of overfields described in Section 2.2 below, and $d = 1$; that is, when $E = F(Y)$ for some curve Y/F .

To begin investigating properties of the groups $\text{III}_{\Omega}^n(E, \mu)$, we first consider some examples:

2.1. Number fields. Let F be a number field, and let Ω be the set of places on F ; the completion of F at $v \in \Omega$ will be denoted by F_v . We recall a few known cases of local-global principles for particular values of d, n, μ . Let ℓ be a prime.

Example 2.1. $d = 0, n = 2, \mu = \mu_\ell$:

As $d = 0$, it follows that E is a finite extension of F . We may identify $H^2(E, \mu_\ell)$ with $\text{Br}(E)[\ell]$. In this case, $\text{III}_\Omega^2(E, \mu_\ell) = 0$ by the Albert-Brauer-Hasse-Noether theorem.

Example 2.2. $d = 1, n = 3, \mu = \mu_\ell^{\otimes 2}$:

Here, writing $E = F(Y)$ for a curve Y/F , Kato shows (in [Kat86, Theorem 0.8(2)]) that $\text{III}_\Omega^3(E, \mu_\ell^{\otimes 2}) = 0$.

Example 2.3. $d = 1, n = 2, \mu = \mu_\ell$:

Write $E = F(Y)$ with Y a smooth projective curve over F , and assume $Y(F) \neq \emptyset$. In this case, $\text{III}_\Omega^2(E, \mu_\ell)$ need not vanish. To see this, first note that $\text{III}_\Omega^2(E, \mu_\ell) \subseteq H^2(E, \mu_\ell) = \text{Br}(E)[\ell]$ and consider the following commutative diagram

$$\begin{array}{ccc} \text{Br}(E)[\ell] & \longrightarrow & \prod_{y \in Y^{(1)}} H^1(\kappa(y), \mu_\ell) \\ \downarrow & & \downarrow \\ \prod_{v \in \Omega} \text{Br}(E_v)[\ell] & \longrightarrow & \prod_v \prod_{z \in Y_{F_v}^{(1)}} H^1(\kappa(z), \mu_\ell). \end{array}$$

If $L = \kappa(y)$ is the residue field of a closed point y of Y , then L/F is a finite extension and any nontrivial element of $H^1(L, \mu_\ell)$ must be nontrivial locally at infinitely many places $v \in \Omega$ by the Grunwald-Wang Theorem (or by application of Chebotarev density). We deduce that any class $\alpha \in \text{III}_\Omega^2(E, \mu_\ell)$ lies in the subgroup $\text{Br}(Y)[\ell] \subseteq \text{Br}(E)[\ell]$ of unramified elements. It follows that

$$\text{III}_\Omega^2(E, \mu_\ell) = \ker(\text{Br}(Y)[\ell] \rightarrow \prod_v \text{Br}(Y_v)[\ell]).$$

Since $Y(F) \neq \emptyset$, we have $\text{Br}(Y) = \text{Br}(F) \oplus H^1(F, \text{Jac}(Y))$ from the Hochschild-Serre spectral sequence. Since $\text{III}_\Omega^2(F, \mu_\ell) = 0$, it follows that $\text{III}_\Omega^2(E, \mu_\ell) = \text{III}_\Omega^1(F, \text{Jac}(Y))[\ell]$ is the ℓ -torsion of the (classical) Tate-Shafarevich group of the Jacobian of Y . In particular, it can be non-trivial; e.g., if Y is the Jacobian of the Selmer curve $3x^3 + 4y^3 + 5z^3$ over $F = \mathbb{Q}$, and $\ell = 3$.

2.2. Curves over complete fields. We now recall some local-global principles in the context of semi-global fields.

Let K be the fraction field of a discrete valuation ring \mathcal{O}_K , and let F be the function field of an absolutely irreducible K -variety. Equivalently, F is a finitely generated field extension of K in which K is algebraically closed. A normal (resp. regular) *model* of F is an integral \mathcal{O}_K -scheme \mathcal{X} with function field F that is flat and projective over \mathcal{O}_K of relative dimension $\text{tr. deg.}_K(F)$, and that is normal (resp. regular) as a scheme. A normal model always exists; e.g., the normalization of a projective closure of an affine \mathcal{O}_K -variety with function field F .

In particular, consider the case of transcendence degree one; i.e., where \mathcal{X} is a relative \mathcal{O}_K -curve. A regular model of F then exists if \mathcal{O}_K is excellent, by the main theorem of [Lip78]. If \mathcal{O}_K is complete then \mathcal{O}_K is excellent, and so there is a regular model of F . In this situation, where \mathcal{O}_K is complete and $\text{tr. deg.}_K(F) = 1$, we call F a *semi-global* field over K .

Let m be a positive integer that is not divisible by the characteristic of K . We consider the following two situations:

2.2.1. *Local-global principle with respect to scheme theoretic points of the closed fiber.* Let Ω be the set of scheme-theoretic points on the closed fiber of \mathcal{X} . For $x \in \Omega$, let $F_x = \text{frac}(\widehat{\mathcal{O}}_{\mathcal{X},x})$ be the fraction field of the complete local ring at x ; this is an overfield of K . If $n \geq 2$, then the associated kernel $\text{III}_{\Omega}^n(F, \mu_m^{\otimes n-1})$ is trivial by [HHK14, Theorem 3.2.3(i)]. On the other hand if $n = 1$, it follows from [HHK15, Corollary 6.5] that $\text{III}_{\Omega}^1(F, \mathbb{Z}/m\mathbb{Z})$ is finite but not necessarily trivial.

2.2.2. *Local-global principle with respect to divisorial discrete valuations.* If K is of equal characteristic, and $\Omega = \mathcal{X}^{(1)}$ is the set of codimension 1 points of \mathcal{X} , then $\text{III}_{\Omega}^n(F, \mu_m^{\otimes n-1}) = 0$, for $n \geq 2$, by [HHK14, Theorem 3.3.6]. This result has now been shown to carry over to assertions in mixed characteristic as well. In particular, see Theorem 4.3 below, which was shown by R. Parimala and V. Suresh, and which we use in the sequel. Other extensions have been announced as well (see [Sak20]). However, the group $\text{III}_{\Omega}^2(F, \mu)$ is not necessarily trivial for μ a finite Galois module (different from $\mu_m^{\otimes n-1}$) and F the function field of a curve over a local field (see [CPS16, Corollaire 5.3(iii)]). Concerning the case of $n = 1$, $\text{III}_{\Omega}^1(F, \mathbb{Z}/m\mathbb{Z})$ is not necessarily trivial (see [CPS12, Section 6]); moreover, it is equal to the corresponding obstruction group in 2.2.1 above, by [HHK15, Proposition 5.1, Corollary 5.3].

3. FUNCTION FIELDS OF RATIONAL CURVES

Let F be a field, and let μ be a finite commutative étale group scheme over F , whose order is prime to the characteristic of F . In Proposition 3.2 below, we relate $\text{III}_{\Omega}^n(F, \mu)$ to $\text{III}_{\Omega}^n(E, \mu)$, where $E = F(t)$ and where Ω is a collection of overfields of F . First we prove a general lemma.

Lemma 3.1. *Let L/F be an arbitrary field extension, and let E be the function field of a smooth, geometrically integral curve Y/F . Let v be a valuation on E corresponding to a closed point P on Y , and let Q be a closed point of $Y_L := Y \times_F L$ above P . Then we have an inclusion of discrete valuation rings $\mathcal{O}_{Y,P} \subseteq \mathcal{O}_{Y_L,Q}$, and the ramification index (i.e., valuation in $\mathcal{O}_{Y_L,Q}$ of a uniformizer of $\mathcal{O}_{Y,P}$) is equal either to 1, or if F has finite characteristic p , to a power of p .*

Proof. First consider the case in which $Y = \mathbb{P}_F^1$, so that $E = F(t)$. The assertion is trivial if P is the point at infinity, so assume that P is a closed point on the affine line. Thus P is given by a monic irreducible polynomial $f(t) \in F[t]$. Let the irreducible factorization of $f(t)$ in $L[t]$ be $\prod_j f_j(t)^{r_j}$ for some distinct monic irreducible polynomials $f_j(t) \in L[t]$ and some non-negative integers r_j . The integers r_j are the ramification indices of the points $Q_j \in Y_L$ that lie over $P \in Y$. We want to show that each r_j either equals 1 or is a power of p if F has characteristic $p > 0$.

Let \bar{L} be an algebraic closure of L , and let \bar{F} be the algebraic closure of F in \bar{L} . Since f is irreducible over F , it factors over $\bar{F}[t]$ as $\prod_i (t - a_i)^{s_i}$ for some distinct elements $a_i \in \bar{F}$ and some non-negative integers s_i that are each equal to 1 or a power of $p = \text{char}(F)$. So this is also the irreducible factorization of $f(t)$ in $\bar{L}[t]$.

For each j , the roots of $f_j(t)$ in \bar{L} are a subset of the roots of f in \bar{L} , viz. the elements a_i . Choose one of these roots a_i of $f_j(t)$, say with multiplicity m_i in f_j . So $f_j(t)$ is the minimal

polynomial of a_i over L , and a_i is not the root of any other f_j . The multiplicity of a_i as a root of $f(t) = \prod_j f_j(t)^{r_j}$ in \bar{L} is thus $m_i r_j$. Thus $m_i r_j = s_i$, and so r_j is equal either to 1 or to a power of p . This completes the proof in the case that $Y = \mathbb{P}_F^1$.

For the general case, since Y is smooth over F , there is a uniformizer on Y at P . This defines a morphism $\pi : Y \rightarrow \mathbb{P}_F^1$ that is étale at P , so that the ramification index of P over the point $P' = (t = 0)$ on \mathbb{P}_F^1 is equal to 1. We then have a pullback diagram

$$\begin{array}{ccc} Y_L & \xrightarrow{\pi'} & \mathbb{P}_L^1 \\ \downarrow \sigma' & & \downarrow \sigma \\ Y & \xrightarrow{\pi} & \mathbb{P}_F^1. \end{array}$$

For any point $Q \in Y_L$ over $P \in Y$, its image Q' in \mathbb{P}_L^1 lies over P' . So by the above special case, the ramification index of σ at Q' is either 1 or a power of p . And since π is étale at Q , its pullback π' is étale at Q' ; and so π' has ramification index 1 there. Thus the ramification index of $\pi\sigma' = \sigma\pi'$ at Q is also of this form. Hence so is the ramification index of σ' at Q , as asserted. \square

Turning to our situation, recall that there is an exact sequence

$$0 \rightarrow H^n(F, \mu) \rightarrow H^n(F(\mathbb{P}^1), \mu) \rightarrow \coprod_{y \in (\mathbb{P}_F^1)^{(1)}} H^{n-1}(\kappa(y), \mu(-1)) \rightarrow H^{n-1}(F, \mu(-1)) \rightarrow 0,$$

where the second map is the sum of the residue maps, and where the map on the right is the sum of the corestriction maps (e.g., see [GMS03, Part 1, Theorem 9.2]; or [GS06, Theorem 6.9.1] in case $\mu = \mu_m^{\otimes r}$). We will refer to this as the *Faddeev sequence*.

Choosing the point at infinity defines a splitting of this rightmost map, and one obtains a short exact sequence:

$$0 \rightarrow H^n(F, \mu) \rightarrow H^n(F(\mathbb{P}^1), \mu) \rightarrow \coprod_{y \in \mathbb{A}_F^{1(1)}} H^{n-1}(\kappa(y), \mu(-1)) \rightarrow 0.$$

We then have the following proposition for a purely transcendental extension E of a field F .

Proposition 3.2. *Let F be a field; let $F_v, v \in \Omega$, be a collection of overfields of F ; and let $E = F(t)$ be a purely transcendental extension of F . Let μ be a finite commutative étale group scheme over F , whose order is prime to the characteristic of F . Take $n \geq 1$ an integer. We have the following exact sequence:*

$$0 \rightarrow \text{III}_{\Omega}^n(F, \mu) \rightarrow \text{III}_{\Omega}^n(E, \mu) \rightarrow \coprod_{y \in \mathbb{A}_F^{1(1)}} \text{III}_{\Omega}^{n-1}(\kappa(y), \mu(-1)) \rightarrow 0.$$

Proof. Applying the splitting of the Faddeev sequence to the fields F and F_v , $v \in \Omega$, we obtain a commutative diagram:

$$(2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^n(F, \mu) & \longrightarrow & H^n(E, \mu) & \xrightarrow{\phi} & \coprod_{y \in (\mathbb{A}_F^1)^{(1)}} H^{n-1}(\kappa(y), \mu(-1)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_v H^n(F_v, \mu) & \longrightarrow & \prod_v H^n(E_v, \mu) & \longrightarrow & \prod_v \coprod_{z \in (\mathbb{A}_{F_v}^1)^{(1)}} H^{n-1}(\kappa(z), \mu(-1)) & \longrightarrow & 0, \end{array}$$

where $E_v = F_v(t)$, and the rightmost vertical map is defined as follows: For a closed point $y \in \mathbb{A}_F^1$, and $v \in \Omega$, we may consider the finite set of closed points of $\mathbb{A}_{F_v}^1$ lying over y . These correspond to the maximal ideals of the ring $\kappa(y) \otimes_F F_v$. For each such point $z \in \mathbb{A}_{F_v}^1$, we have an extension of local rings $\mathcal{O}_{\mathbb{A}_{F_v}^1, z} / \mathcal{O}_{\mathbb{A}_F^1, y}$. We set e_z to be the ramification index of this extension. We then define the rightmost vertical map to be the sum of the maps $H^{n-1}(\kappa(y), \mu(-1)) \rightarrow H^{n-1}(\kappa(z), \mu(-1))$ given by the natural pullback and multiplication by e_z . The fact that the diagram (2) commutes follows from [GMS03, Proposition 8.2]. By Lemma 3.1, each ramification degree e_z is a power of the characteristic, and since this is prime to the order of μ , it follows that the kernel of this map is the same as the kernel of the natural pullback maps.

From this, we deduce a short exact sequence of kernels

$$0 \rightarrow \mathrm{III}_{\Omega}^n(F, \mu) \rightarrow \mathrm{III}_{\Omega}^n(E, \mu) \rightarrow \prod_{y \in \mathbb{A}_F^1(1)} \mathrm{III}_{\Omega}^{n-1}(\kappa(y), \mu(-1)).$$

It remains to show that the rightmost map is surjective.

Choose $\beta \in \prod \mathrm{III}_{\Omega}^{n-1}(\kappa(y), \mu(-1))$ and $\alpha \in H^n(E, \mu)$ with $\phi(\alpha) = \beta$. Choose $p \in \mathbb{A}^1(F)$ a rational point disjoint from the support of β . In particular, α is unramified at p . Put $\alpha' = \alpha - \alpha(p)_E$, where $\alpha(p)$ is the specialization of α at p and $\alpha(p)_E$ is the pullback of the class $\alpha(p)$ via the inclusion $F \subset E$. Note that $\phi(\alpha') = \beta$. We claim that $\alpha' \in \mathrm{III}_{\Omega}^n(E, \mu)$. Let α'_{E_v} be the image of α' in $H^n(E_v, \mu)$. By the commutativity of the diagram (2), α'_{E_v} maps to zero in $\prod_{z \in (\mathbb{A}_{F_v}^1)^{(1)}} H^{n-1}(\kappa(z), \mu(-1))$, hence α' comes from a class in $H^n(F_v, \mu)$. In particular, it is determined by the value of its specialization at any F_v -rational point. But its specialization at p is $\alpha'(p) = \alpha(p) - \alpha(p) = 0$, so that $\alpha'_{E_v} = 0$ and $\alpha' \in \mathrm{III}_{\Omega}^n(E, \mu)$. \square

4. CURVES OVER SEMI-GLOBAL FIELDS

Let F be a semi-global field over a complete discretely valued field K ; i.e., the function field of a projective K -curve (as always, assumed absolutely irreducible). Let E be the function field of a projective curve Y over F . In this situation, we can consider local-global principles for E with respect to the valuations on F , or with respect to those on E . We relate the corresponding III groups in Propositions 4.6 and 4.8 below.

In this section we will be interested in the following sets of valuations. Let Ω_F be the set of divisorial discrete valuations on F ; i.e. valuations corresponding to codimension one points of some normal model of F over the valuation ring \mathcal{O}_K of K . In fact, a regular model \mathcal{X} of F exists, as noted in Section 2.2; and we write $\Omega_{\mathcal{X}} \subseteq \Omega_F$ for the set of valuations on F corresponding to codimension one points on \mathcal{X} . For $v \in \Omega_F$, let F_v be the completion of F at v and let $E_v = E \otimes_F F_v$. Similarly, we let Ω_E be the set of divisorial discrete valuations on E , so that any $v \in \Omega_E$ corresponds to a codimension one point on some normal model of E over \mathcal{O}_K . If \mathcal{Y} is a fixed such normal model of E , we write $\Omega_{\mathcal{Y}} \subseteq \Omega_E$ for the set of valuations on E corresponding to codimension one points on \mathcal{Y} .

Notation 4.1. For the rest of this paper we fix an integer $m > 1$ prime to the residue characteristic of K , and we let ζ_m be a primitive m -th root of unity in an algebraic closure of K . For any field L containing K and any integer $n \geq 0$, we write $H^n(L) := H^n(L, \mu_m^{\otimes(n-1)})$.

With E as above, we write

$$H_{\text{nr}}^n(E) = \bigcap_{v \in \Omega_E} \ker[\partial^n : H^n(E) \rightarrow H^{n-1}(\kappa(v))]$$

for the n -th unramified cohomology of E with coefficients in $\mu_m^{\otimes(n-1)}$. For a normal proper model \mathcal{Y} of E over \mathcal{O}_K we define

$$H_{\text{nr}}^n(E/\mathcal{Y}) = \bigcap_{v \in \Omega_{\mathcal{Y}}} \ker[\partial^n : H^n(E) \rightarrow H^{n-1}(\kappa(v))],$$

which by definition contains $H_{\text{nr}}^n(E)$. We also consider the following III groups:

$$\begin{aligned} \text{III}_E^n(E) &= \text{III}_{\Omega_E}^n(E, \mu_m^{\otimes(n-1)}) = \ker[H^n(E) \rightarrow \prod_{v \in \Omega_E} H^n(E_v)], \\ \text{III}_F^n(E) &= \text{III}_{\Omega_F}^n(E, \mu_m^{\otimes(n-1)}) = \ker[H^n(E) \rightarrow \prod_{v \in \Omega_F} H^n(E_v)], \\ \text{III}_F^n(F) &= \text{III}_{\Omega_F}^n(F, \mu_m^{\otimes(n-1)}) = \ker[H^n(F) \rightarrow \prod_{v \in \Omega_F} H^n(F_v)]. \end{aligned}$$

We have that $\text{III}_E^n(E) \subseteq H_{\text{nr}}^n(E)$, since the residue map factors through the completion.

In this and the following section, we will want to use that $\text{III}_F^n(F)$ is trivial for $n > 1$. In Theorem 3.3.6 of [HHK14], this triviality was shown to hold if the underlying complete discretely valued field K is equicharacteristic; and moreover that $\text{III}_{\Omega_F}^n(F, \mu_m^{\otimes r})$ vanishes for all r if $[F(\zeta_m) : F]$ is prime to m . (In fact, it was shown there that for any regular model \mathcal{X} of F , the corresponding *a priori* larger obstruction groups with respect to $\Omega_{\mathcal{X}}$ vanish.) Several years ago, R. Parimala and V. Suresh proved that if $[F(\zeta_m) : F]$ is prime to m then $\text{III}_{\Omega_F}^n(F, \mu_m^{\otimes r})$ vanishes for all r , even if K is not assumed to be equicharacteristic. Their argument has been unpublished until now (although it has been cited in the literature; e.g. [Hu17, p. 4372]); and we thank them for allowing us to provide it here, in Lemma 4.2 and Theorem 4.3 below.

Lemma 4.2 (Parimala-Suresh). *Let A be a complete two dimensional local ring with maximal ideal $\mathfrak{m} = (\pi, \delta)$, residue field κ and field of fractions L . Let m be a positive integer that is invertible in A . Let $\xi = \sum (a_{i1}) \cdot (a_{i2}) \cdots (a_{im}) \in H^n(L, \mu_m^{\otimes n})$ with $a_{ij} \in A$ representing $(a_{ij}) \in L^\times / (L^\times)^m = H^1(L, \mu_m)$. Suppose that π and δ are the only primes that divide a_{ij} for all i, j . If the image of ξ is zero over L_π and L_δ , then ξ is zero.*

Proof. Since each a_{ij} is of the form $u\pi^\epsilon\delta^{\epsilon'}$ for some unit u in A , $0 \leq \epsilon, \epsilon' < m$ and $(ab) \cdot (c) = (a) \cdot (c) + (b) \cdot (c)$ and $(a) \cdot (-a) = 0$, it follows that

$$\xi = \xi_0 + \xi_1 \cdot (\pi) + \xi_2 \cdot (\delta) + \xi_3 \cdot (\pi) \cdot (\delta),$$

where ξ_0 is a sum of symbols of the form $(t_1) \cdot (t_2) \cdots (t_n)$; ξ_1, ξ_2 are sums of symbols of the form $(u_1) \cdot (u_2) \cdots (u_{n-1})$; and ξ_3 is a sum of symbols $(w_1) \cdots (w_{n-2})$, with t_i, u_j, w_s units in A . Since m is a unit in A , we have $\xi_0 \in H_{\text{et}}^n(A, \mu_m^{\otimes n})$; $\xi_1, \xi_2 \in H_{\text{et}}^{n-1}(A, \mu_m^{\otimes(n-1)})$; and $\xi_3 \in H_{\text{et}}^{n-2}(A, \mu_m^{\otimes(n-2)})$. Let $\partial_\pi : H^n(L, \mu_m^{\otimes n}) \rightarrow H^{n-1}(\kappa(\pi), \mu_m^{\otimes(n-1)})$ be the residue homomorphism. Then $\partial_\pi(\xi) = \bar{\xi}_1 - \bar{\xi}_3 \cdot (\bar{\delta})$, where bar denotes the image in $A/(\pi)$. The image of ξ in $H^n(L_\pi, \mu_m^{\otimes n})$ is zero, and so $\partial_\pi(\xi) = \bar{\xi}_1 - \bar{\xi}_3 \cdot (\bar{\delta}) = 0$. Since $A/(\pi)$ is a complete discrete valuation ring with field of fractions $\kappa(\pi)$ and residue field κ , we have the

residue homomorphism $\partial_{\bar{\delta}} : H^{n-1}(\kappa(\pi), \mu_m^{\otimes n-1}) \rightarrow H^{n-2}(\kappa, \mu_m^{\otimes n-2})$. Here $\bar{\delta}$ is a parameter in $A/(\pi)$; so $\partial_{\bar{\delta}}(\bar{\xi}_1) = 0$ and $\partial_{\bar{\delta}}(\bar{\xi}_3 \cdot (\bar{\delta})) = \bar{\xi}_3$, where $\bar{\xi}_3$ denotes the image in $A/\mathfrak{m} = \kappa$. But $\bar{\xi}_1 - \bar{\xi}_3 \cdot (\bar{\delta}) = 0$, so $\bar{\xi}_3 = \partial_{\bar{\delta}}(\bar{\xi}_1 - \bar{\xi}_3 \cdot (\bar{\delta})) = 0$. Since A is a complete regular local ring, we have $\xi_3 = 0$. The element $\bar{\xi}_1 - \bar{\xi}_3 \cdot (\bar{\delta})$ is zero in $H^{n-1}(\kappa(\pi), \mu_m^{\otimes n-1})$, and so $\bar{\xi}_1$ is zero in $H^{n-1}(\kappa(\pi), \mu_m^{\otimes n-1})$. Since $A/(\pi)$ is a complete discrete valuation ring with field of fractions $\kappa(\pi)$, the map $H_{\text{et}}^{n-1}(A/(\pi), \mu_m^{\otimes n-1}) \rightarrow H^{n-1}(\kappa(\pi), \mu_m^{\otimes n-1})$ is injective and hence $\bar{\xi}_1$ is zero in $H_{\text{et}}^{n-1}(A/(\pi), \mu_m^{\otimes n-1})$. Since π is in the maximal ideal of the complete local ring A , it follows that ξ_1 is zero. Similarly, using that the image of ξ is zero in L_{δ} , it follows that ξ_2 is zero. Finally, since ξ_0 is zero over L_{π} , we deduce that $\bar{\xi}_0$ is zero in $H_{\text{et}}^n(A/(\pi), \mu_m^{\otimes n}) \hookrightarrow H^n(\kappa(\pi), \mu_m^{\otimes n})$, so that, similarly to the above, $\xi_0 = 0$. Hence $\xi = 0$. \square

Theorem 4.3 (Parimala-Suresh). *Let K be a complete discretely valued field with residue field κ , and let F be the function field of a regular curve over K . Let m be a positive integer that is not divisible by the characteristic of κ , and assume that $[F(\zeta_m) : F]$ is prime to m . Then $\text{III}_F^n(F, \mu_m^{\otimes r}), n \geq 2$, is trivial for every integer r . Equivalently, the natural homomorphism*

$$\iota : H^n(F, \mu_m^{\otimes r}) \rightarrow \prod_{v \in \Omega_F} H^n(F_v, \mu_m^{\otimes r})$$

is injective for $n \geq 2$.

Proof. First consider the case where $r = n$, and let $\xi \in H^n(F, \mu_m^{\otimes n})$. It follows from the Bloch-Kato conjecture [Voe11] that

$$\xi = \sum_i (a_{i1}) \cdot \dots \cdot (a_{in}),$$

where $(a_{ij}) \in H^1(F, \mu_m)$ for some $a_{ij} \in F^\times$. Let \mathcal{X} be a regular proper model of F over the valuation ring R of K such that $\cup \text{Supp}_{\mathcal{X}}(a_{ij})$ is a union of regular curves with normal crossings. Such a regular model exists by [Lip75, page 193] and [Lip78].

Suppose that the image of ξ in each $H^n(F_v, \mu_m^{\otimes n})$ is zero. Let P be a closed point of \mathcal{X} (thus lying on the closed fiber), and let $A = \hat{\mathcal{O}}_{\mathcal{X}, P}$ be the completion of the local ring of \mathcal{X} at P . By the normal crossings hypothesis, we may apply Lemma 4.2 to A for suitable elements π, δ ; and so the image of ξ in $H^n(F_P, \mu_m^{\otimes n})$ is zero. Since $[F(\zeta_m) : F]$ is prime to m , it follows from [HHK14, Theorem 3.2.3(i)] that $\xi = 0$, proving this case.

For the case of a general value of r , let $\xi \in H^n(F, \mu_m^{\otimes r})$ be in the kernel of ι . Since $H^n(F(\zeta_m), \mu_m^{\otimes r}) \simeq H^n(F(\zeta_m), \mu_m) \simeq H^n(F(\zeta_m), \mu_m^{\otimes n})$, the image of ξ in $H^n(F(\zeta_m), \mu_m^{\otimes r})$ is zero, by the above special case. Since $[F(\zeta_m) : F]$ is prime to m , it follows that $\xi = 0$. \square

Of course the hypothesis that $[F(\zeta_m) : F]$ is prime to m is automatic if m is prime. The argument below shows that we can always make the latter assumption in order to establish that $\text{III}_F^n(F, \mu_m^{\otimes n-1})$ is trivial. We thank Jean-Louis Colliot-Thélène for pointing that out.

Corollary 4.4. *Let K be a complete discretely valued field with residue field κ , and let F be the function field of a regular curve over K . Let m be a positive integer that is not divisible by the characteristic of κ . Then $\text{III}_F^n(F, \mu_m^{\otimes n-1}), n \geq 2$, is trivial.*

Proof. By Theorem 4.3 above, the assumption holds if m is prime. In the general case, by decomposing m into a product of prime powers, we may assume that $m = \ell^r$ is a power of

a prime number. Using the Bloch-Kato conjecture [Voe11], one has that the map

$$H^n(F, \mu_{\ell^r}^{\otimes n-1}) \xrightarrow{\iota_r} H^n(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))$$

is injective. (See the argument that was given in [MS82, 18.4(c)] in the case $n = 3$. Namely, since $\ker \iota_r \simeq H^{n-1}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))/\ell^r$ by the Kummer exact sequence, it suffices to show that $H^{n-1}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))$ is ℓ -divisible. By Bloch-Kato this is isomorphic to $\varinjlim_i K_{n-1}^M F/\ell^i$, where the transition maps are multiplication by ℓ ; and this is ℓ -divisible.) Hence the group $H^n(F, \mu_{\ell^r}^{\otimes n-1})$ is the ℓ^r -torsion subgroup of the group $H^n(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))$. By Theorem 4.3 above (applied to the group $\text{III}_F^n(F, \mu_\ell^{\otimes n-1})$), the map

$$H^n(F, \mu_\ell^{\otimes n-1}) \rightarrow \prod_{v \in \Omega_F} H^n(F_v, \mu_\ell^{\otimes n-1})$$

is injective; hence the kernel of the map

$$H^n(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1)) \rightarrow \prod_{v \in \Omega_F} H^n(F_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))$$

has no ℓ -torsion. Hence it has no ℓ^r -torsion. This gives that the map

$$H^n(F, \mu_{\ell^r}^{\otimes n-1}) \rightarrow \prod_{v \in \Omega_F} H^n(F_v, \mu_{\ell^r}^{\otimes n-1})$$

is injective as well, as desired. \square

Lemma 4.5. *Consider a morphism of integral Noetherian schemes $f : V \rightarrow W$, and let v be a discrete valuation on the function field $\kappa(V)$ corresponding to a prime divisor $D \subset V$. If the restriction w of v to the function field $\kappa(W)$ of W is nontrivial, then there is a commutative diagram*

$$\begin{array}{ccc} \tilde{V} & \longrightarrow & V \\ \downarrow & & \downarrow \\ \tilde{W} & \longrightarrow & W \end{array}$$

whose horizontal maps are birational morphisms, together with a prime divisor \tilde{D} on \tilde{V} that induces the valuation v , and whose image in \tilde{W} is a prime divisor \tilde{E} giving rise to w .

That is, the restriction of a divisorial valuation, if nontrivial, is also divisorial.

Proof. Let d be the dimension of the generic fiber of f . By [Sta19, Lemma 087E], we may find a blowup along some ideal sheaf $\tilde{W} \rightarrow W$ such that if we take the strict transform \tilde{V} of V with respect to the blowup, the morphism $\tilde{V} \rightarrow \tilde{W}$ is flat. Note that we also have $\tilde{V} \rightarrow V$ is a blowup along the corresponding inverse ideal sheaf by [Sta19, Lemma 080E], and therefore is a birational morphism of integral Noetherian schemes. Since $\tilde{V} \rightarrow V$ is proper, by the valuative criterion, we may find a divisor \tilde{D} of \tilde{V} giving rise to v , whose image in V is D . Now, consider the image $\tilde{\Delta}$ of \tilde{D} in \tilde{W} . Since the valuation v is nontrivial on $\kappa(W)$, it follows that the morphism $\tilde{D} \rightarrow W$ is not dominant; hence neither is $\tilde{D} \rightarrow \tilde{W}$, and so $\tilde{\Delta} \neq \tilde{W}$. Since $\tilde{V} \rightarrow \tilde{W}$ is flat of relative dimension d , we have $\dim(\tilde{\Delta} \times_{\tilde{W}} \tilde{V}) = \dim \tilde{\Delta} + d \leq \dim W + d - 1$. But since $\tilde{D} \subseteq \tilde{\Delta} \times_{\tilde{W}} \tilde{V}$, and $\dim \tilde{D} = \dim V - 1 = \dim W + d - 1$, it follows

that $\dim(\tilde{\Delta} \times_{\tilde{W}} \tilde{V}) = \dim W + d - 1$ and hence $\dim \tilde{\Delta} = \dim W - 1$. In particular $\tilde{\Delta}$ is a divisor and since the center of w must contain $\tilde{\Delta}$, it must be equal to $\tilde{\Delta}$ as desired. \square

Proposition 4.6. *Let E be the function field of a regular projective curve over a semi-global field F . Then*

$$\mathbb{H}_F^n(E) \subseteq \mathbb{H}_E^n(E) \subseteq H_{\text{nr}}^n(E), \text{ for all } n \geq 3.$$

Proof. The second containment was observed following Notation 4.1. For the first containment, let $\alpha \in \mathbb{H}_F^n(E)$. Consider $w \in \Omega_E$, corresponding to a codimension one point y on some normal model \mathcal{Y} of E . We wish to show that the image α_y of α in $H^n(E_w)$ vanishes. Let $D_y \subset \mathcal{Y}$ be the closure of y in \mathcal{Y} . Let \mathcal{X} be a regular model of F . Note that the rational map $\mathcal{Y} \dashrightarrow \mathcal{X}$ extends to a regular map $\mathcal{U} \rightarrow \mathcal{X}$ on some open set $\mathcal{U} \subseteq \mathcal{Y}$ containing y .

Assume first that D_y is a vertical divisor; i.e., y is in a fiber of π over $x \in \mathcal{X}$. By Lemma 4.5, after changing the model \mathcal{X} we may assume that x is a codimension 1 point of \mathcal{X} , and y is the generic point of one of the components of the closed fiber of $\mathcal{U} \times_{\mathcal{X}} \mathcal{O}_{\mathcal{X},x}$. Let $v = v_x \in \Omega_F$ be the corresponding divisorial valuation. Then y is also the generic point of one of the components of the closed fiber of $\mathcal{U} \times_{\mathcal{X}} \hat{\mathcal{O}}_{\mathcal{X},x}$, so that the map $H^n(E) \rightarrow H^n(E_w)$ factors through $H^n(E_v)$. Since α maps to zero in $H^n(E_v)$, we obtain that the image of α in $H^n(E_w)$ is zero.

Assume next that D_y is a horizontal divisor; i.e., the restriction to F of the discrete valuation corresponding to y is trivial. Thus we can view y as a codimension one point of the generic fiber Y of \mathcal{Y} over F , whose residue field $L = \kappa(D_y)$ is a finite extension of F . Let v_z be a place on L corresponding to a codimension 1 point z on some regular proper model $\tilde{D}_y \rightarrow \mathcal{X}$ of L , and let x be the image of z in \mathcal{X} . Again by Lemma 4.5, after changing the model \mathcal{X} along with \tilde{D}_y , we may assume that x is a codimension 1 point of \mathcal{X} . Let $v = v_x$ be the corresponding valuation on F . Let L_z be the completion of L at v_z . Then the natural inclusion $L \subset L_z$ factors through $L \otimes_F F_v$ (which is the direct sum of the residue fields at the closed points of Y_{F_v} over y).

We have the following commutative diagram (similar to [Kat86, p.167]):

$$\begin{array}{ccc} H^n(E) & \longrightarrow & \prod_{y \in Y^{(1)}} H^{n-1}(\kappa(y)) \\ \downarrow j & & \downarrow j' \\ \prod_{v \in \Omega_F} H^n(E_v) & \longrightarrow & \prod_v \prod_{y' \in Y_{F_v}^{(1)}} H^{n-1}(\kappa(y')). \end{array}$$

By assumption, α maps to zero over E_v . Hence the residue $\partial_{v_y}^n(\alpha)$ maps to zero in $\prod_{y' \in Y_{F_v}^{(1)}} H^{n-1}(\kappa(y'))$, hence is zero over any completion $\kappa(y)_z$ of $\kappa(y)$ as in the discussion above.

Using that $n - 1 \geq 2$, by Corollary 4.4 we deduce that $\partial_{v_y}^n(\alpha) = 0$. Hence by [GMS03, Part 1, II.7.9] and [AGV73, XII, Cor. 5.5(iii)], the image α_y of α in $H^n(E_w)$ satisfies

$$\alpha_y \in \text{im}[H^n(\hat{\mathcal{O}}_{\mathcal{Y},y}) \hookrightarrow H^n(E_w)].$$

Let β be the image of α_y under the natural isomorphism $H^n(\hat{\mathcal{O}}_{\mathcal{Y},y}) \xrightarrow{\sim} H^n(\kappa(y))$ given in [AGV73, XII Cor 5.5(iii)]. Similarly as above, since α maps to zero in $\prod_{v \in \Omega_F} H^n(E_v)$ we

deduce that β satisfies

$$\beta \in \ker[H^n(\kappa(y)) \rightarrow \prod_v \prod_{y' \in Y_{F_v}^{(1)}} H^n(\kappa(D_{y'}))].$$

So $\beta = 0$ by Corollary 4.4 again, and the image α_y of α in $H^n(E_w)$ is indeed zero. \square

On the other hand, Proposition 4.6 does not hold in general for $n = 2$; see Remark 5.4 below.

Lemma 4.7. *Let E be the function field of a regular projective curve over a semi-global field F . Assume that $[F(\zeta_m) : F]$ is prime to m . Then*

$$\mathbb{III}_F^n(E) \supseteq \mathbb{III}_E^n(E), \text{ for all } n \geq 2.$$

Proof. We first observe that it suffices to prove the lemma in the special case that $\zeta_m \in F$. To see this, assume that the result holds in this special case, and in the general case let $\alpha \in \mathbb{III}_E^n(E) \subseteq H^n(E)$. The restriction β of α to $H^n(E(\zeta_m))$ lies in $\mathbb{III}_{E(\zeta_m)}^n(E(\zeta_m))$, and so by the special case it follows that $\beta \in \mathbb{III}_{F(\zeta_m)}^n(E(\zeta_m))$. That is, for every valuation $v' \in \Omega_{F(\zeta_m)}$ lying over a valuation $v \in \Omega_F$, the image $\beta_{v'}$ of β in $H^n(E(\zeta_m)_{v'})$ is trivial. Here $\beta_{v'}$ is the restriction of α_v to $H^n(E(\zeta_m)_{v'})$, where α_v is the image of α in $H^n(E_v)$. But $[E(\zeta_m)_{v'} : E_v]$ is prime to m , so this restriction map is injective (by restriction-corestriction). Hence each α_v is trivial; i.e., α lies in $\mathbb{III}_F^n(E)$, as asserted.

So for the remainder of the proof we assume that $\zeta_m \in F$, and we identify $H^n(E)$ with $H^n(E, \mu_m^{\otimes n})$. Let $\alpha \in \mathbb{III}_E^n(E) \subseteq H^n(E) = H^n(E, \mu_m^{\otimes n})$. Using the Bloch-Kato conjecture [Voe11] (as in the proof of Theorem 4.3 above), we can write $\alpha = \sum_i (a_{i1}) \cdot \dots \cdot (a_{in})$, where $(a_{ij}) \in H^1(E, \mu_m)$, for some $a_{ij} \in E^\times$. By definition, α maps to zero in $H^n(E_w, \mu_m^{\otimes n})$ for every $w \in \Omega_E$; and we wish to show that it maps to zero in $H^n(E_v, \mu_m^{\otimes n})$ for every $v \in \Omega_F$.

Take $v \in \Omega_F$, corresponding to a codimension one point x on a regular model \mathcal{X} of F . The local ring $\mathcal{O}_{\mathcal{X},x}$ is a discrete valuation ring. Let $\mathcal{Y}_x \rightarrow \mathcal{O}_{\mathcal{X},x}$ be a regular model of E over $\mathcal{O}_{\mathcal{X},x}$ such that $\cup \text{Supp}(a_{ij})$ is a union of regular curves with normal crossings on \mathcal{Y}_x . Such a model exists by [Lip75, page 193], since \mathcal{O}_K and hence $\mathcal{O}_{\mathcal{X},x}$ is excellent. Thus $\hat{\mathcal{Y}}_x := \mathcal{Y}_x \times_{\mathcal{O}_{\mathcal{X},x}} \hat{\mathcal{O}}_{\mathcal{X},x}$ is also regular, and $\cup \text{Supp}(a_{ij})$ is a union of regular curves with normal crossings on $\hat{\mathcal{Y}}_x$ as well. Here $E_v = F_v(Y)$, where Y is the generic fiber of \mathcal{Y}_x over F ; and this is a semi-global field over F_v that contains E .

We need to prove that the image of α in $H^n(F_v(Y)) = H^n(F_v(Y), \mu_m^{\otimes n})$ is zero. Since $n \geq 2$, we may apply [HHK14, Theorem 3.2.3(i)] to the curve $\hat{\mathcal{Y}}_x$ over the complete discrete valuation ring $\hat{\mathcal{O}}_{\mathcal{X},x}$. By that result, it is enough to prove that for every point P on the closed fiber \hat{Z} of $\hat{\mathcal{Y}}_x$, the image of α in $H^n(F_v(Y)_P, \mu_m^{\otimes n})$ is zero. (Here $F_v(Y)_P$ is the fraction field of the complete local ring of $\hat{\mathcal{Y}}_x$ at the point P .)

We may identify the closed fiber Z of \mathcal{Y}_x with \hat{Z} , and thereby regard the above point $P \in \hat{Z}$ as a point of $Z \subset \mathcal{Y}_x$. Then $F_v(Y)_P$ is identified with the fraction field E_P of the complete local ring $\hat{\mathcal{O}}_{\mathcal{Y}_x,P}$ of \mathcal{Y}_x at P . If P is a codimension one point of \mathcal{Y}_x , then it corresponds to some $w \in \Omega_E$. Thus $\alpha \in \mathbb{III}_E^n(E)$ becomes zero over $E_w = E_P$, as needed. On the other hand, if P is a codimension two point of \mathcal{Y}_x , then take a system of local parameters (π, δ) at $P \in \mathcal{Y}_x$, with π, δ each corresponding to an element of Ω_E . By the above condition on $\cup \text{Supp}(a_{ij})$, we may choose these parameters so that π and δ are the only primes of

$\hat{\mathcal{O}}_{\mathcal{Y}_{x,P}}$ that divide all the a_{ij} . Since $\alpha \in \text{III}_E^n(E)$ becomes zero over $E_\pi \subseteq (E_P)_\pi$ and over $E_\delta \subseteq (E_P)_\delta$, Lemma 4.2 implies that α becomes zero over $E_P = F_v(Y)_P$, completing the proof. \square

Combining Proposition 4.6 and Lemma 4.7, we obtain:

Proposition 4.8. *Let E be the function field of a projective curve over a semi-global field F . Assume that $[F(\zeta_m) : F]$ is prime to m . Then $\text{III}_F^n(E) \supseteq \text{III}_E^n(E)$ for all $n \geq 2$, and $\text{III}_F^n(E) = \text{III}_E^n(E) \subseteq H_{\text{nr}}^n(E)$ for all $n \geq 3$.*

5. HIGHER LOCAL-GLOBAL PRINCIPLES

5.1. Local-global principles with respect to valuations on F . We now turn to the question of when local-global principles hold for a function field E over a semi-global field F ; i.e., when the corresponding obstruction groups $\text{III}_F^n(E)$ vanish. (Here and below, we preserve Notation 4.1, for our fixed value of $m > 1$.) Using results of the previous section, we treat the case where $E = F(\mathbb{P}_E^1)$ in Theorem 5.2 below; and we consider the situation where the ground field K is a local field in Theorem 5.5. First we state some preliminaries.

If \mathcal{X} is a regular model of a semi-global field over \mathcal{O}_K , then the associated *reduction graph* was defined in [HHK15, Section 6], encoding the configuration of intersections of the irreducible components of the closed fiber. In the case that the model \mathcal{X} has semi-stable reduction, this graph is homotopy equivalent to the dual graph defined in [DM69, p. 86] (and is in fact its barycentric subdivision; see [HHK15, Remark 6.1]).

Lemma 5.1. *If F is a semi-global field, then there is a finite separable extension F' of F having a regular model whose reduction graph is not a tree.*

Proof. It suffices to prove that there exists a normal model \mathcal{X} of F' with this property, since there is then a desingularization $\widetilde{\mathcal{X}} \rightarrow \mathcal{X}$ by the main theorem in [Lip78]; and its reduction graph, which maps onto that of \mathcal{X} , is then also not a tree.

First consider the case that $F = K(x)$, with a model $\mathcal{X} = \mathbb{P}_{\mathcal{O}_K}^1$. Pick an integer $r > 1$ that is prime to the residue characteristic of \mathcal{O}_K . Let t be a uniformizer of \mathcal{O}_K , let $f, g \in \mathcal{O}_K[x]$ be irreducible monic polynomials whose loci in \mathcal{X} are disjoint, and let \mathcal{X}' be the normalization of \mathcal{X} in the finite extension $F' := F[z]/(z^n - (f^n - t)(g^n - t))$. Then \mathcal{X}' is a normal model of F' , and the reduction graph of \mathcal{X}' is not a tree.

For the general case, we may write F as a finite extension of a purely transcendental field $F_0 := K(x)$. Let F_1 be the maximal separable subextension of F/F_0 ; thus F/F_1 is purely inseparable. Since purely inseparable morphisms induce homeomorphisms on the underlying topological spaces, it suffices to prove the lemma for F_1 ; and so we may assume that the finite extension F/F_0 is separable. Here $\mathcal{X}_0 := \mathbb{P}_{\mathcal{O}_K}^1$ is a regular model of $K(x)$ over \mathcal{O}_K , and the normalization \mathcal{X} of \mathcal{X}_0 in F is a normal model of F , which is equipped with a finite generically separable morphism $\phi : \mathcal{X} \rightarrow \mathcal{X}_0$. Pick two distinct closed points P, Q on the closed fiber of \mathcal{X}_0 away from the point at infinity, over which \mathcal{X} is regular, and which do not lie in the closure of any point in the generic fiber of \mathcal{X}_0 at which ϕ is branched. The points $P, Q \in \mathbb{A}_{\mathcal{O}_K}^1 \subset \mathbb{P}_{\mathcal{O}_K}^1$ are given by maximal ideals (f, t) and (g, t) in $\mathcal{O}_K[x]$, such that $f, g \in \mathcal{O}_K[x]$ are irreducible monic polynomials having disjoint loci in \mathcal{X}_0 . Choose $r > 1$ relatively prime to the residue characteristic of \mathcal{O}_K . Let $\mathcal{X}'_0 \rightarrow \mathcal{X}_0 = \mathbb{P}_{\mathcal{O}_K}^1$ be the branched cover given in the previous paragraph, and let \mathcal{X}' be the normalization of $\mathcal{X}'_0 \times_{\mathcal{X}_0} \mathcal{X}$. The reduction graph of \mathcal{X}' is not a tree, and its function field L is then as asserted. \square

Theorem 5.2. *Let F be a semi-global field over a complete discretely valued field K . Let $E = F(t)$, the function field of \mathbb{P}_F^1 . Then:*

- (a) $\mathbb{III}_F^1(E)$ is trivial if and only if the reduction graph of a regular model \mathcal{X} of F over \mathcal{O}_K is a tree;
- (b) $\mathbb{III}_F^2(E)$ is non-trivial;
- (c) $\mathbb{III}_F^n(E)$ is trivial for $n > 2$.

Proof. Assume first that $n > 2$. Since $n \geq 3$, $\mathbb{III}_F^n(E) \subseteq H_{\text{nr}}^n(E)$ by Proposition 4.6. So by Proposition 3.2, each element of $\mathbb{III}_F^n(E)$ is in the image of $\mathbb{III}_F^n(F) = \mathbb{III}_{\Omega_F}^n(F, \mu_m^{\otimes n-1})$. By Corollary 4.4, this last group is zero. This gives part (c).

Assume now that $n = 2$. By Lemma 5.1, there is a finite extension L/F such that the reduction graph of a model of L is not a tree. Let $y \in Y = \mathbb{P}_F^1$ be a closed point with residue field L . Then in the right term of the sequence in Proposition 3.2, for $\kappa(y) = L$ we have that $\mathbb{III}_F^1(L) \neq 0$ (see [HHK15, Proposition 5.1, Corollary 5.3 and Corollary 6.5]). Hence $\mathbb{III}_F^2(E) \neq 0$ as well, by Proposition 3.2, proving part (b).

Finally, in the case $n = 1$, the last term in the exact sequence in Proposition 3.2 is trivial since \mathbb{III}^0 is trivial. So by that result, $\mathbb{III}_F^n(F) = \mathbb{III}_F^n(E)$. The former group is trivial if and only if the reduction graph is a tree, by the results from [HHK15] cited above in the $n = 2$ case. So assertion (a) follows. \square

Remark 5.3. One has a similar result to part (c) above: $\mathbb{III}_{\Omega}^n(E) := \mathbb{III}_{\Omega}^n(E, \mu_m^{\otimes n-1}) = \ker[H^n(E) \rightarrow \prod_{x \in \Omega} H^n(E_x, \mu_m^{\otimes n-1})]$ is trivial for $n > 2$, where Ω is the set of points on the closed fiber of \mathcal{X} ; $F_x = \text{frac}(\widehat{\mathcal{O}_{\mathcal{X},x}})$ for $x \in \Omega$; and $E_x = \text{frac}(E \otimes_F F_x)$. Namely, as before we have $\mathbb{III}_{\Omega}^n(E) \subseteq H_{\text{nr}}^n(E)$, and so $\mathbb{III}_{\Omega}^n(E)$ is contained in the image of $\mathbb{III}_{\Omega}^n(F) := \mathbb{III}_{\Omega}^n(F, \mu_m^{\otimes n-1})$, which vanishes by Section 2.2.1.

Remark 5.4. Theorem 5.2 shows that Proposition 4.6 does not hold in general for $n = 2$. Namely, take $K = \mathbb{C}((t))$, $F = k(x)$, and $E = F(y)$, with $\mathcal{X} = \mathbb{P}_{\mathbb{C}[[t]]}^1$ and $\mathcal{Y} = \mathbb{P}_{\mathcal{X}}^1$. Then $\mathbb{III}_F^2(E)$ is non-trivial by Theorem 5.2(b). But we claim that $\mathbb{III}_{\mathcal{Y}}^2(E)$ is trivial and hence so is $\mathbb{III}_E^2(E)$. To see the triviality of $\mathbb{III}_{\mathcal{Y}}^2(E)$, recall that $\mathbb{III}_{\mathcal{Y}}^2(E) \subseteq H_{\text{nr}}^2(E)$. Since the unramified cohomology group $H_{\text{nr}}^2(-)$ is a stable birational invariant, it satisfies the conditions of specialization and homotopy (see [CT80, p. 216]), and the natural map $H^2(k) \rightarrow H_{\text{nr}}^2(E)$ is an isomorphism by [CT80, Théorème 1.5]. (This conclusion can also be obtained by the Faddeev sequence and a diagram chase.) But since $k = \mathbb{C}((t))$ is a C_1 field, we have $H^2(k) = 0$, showing that $H_{\text{nr}}^2(E) = 0$, and hence $\mathbb{III}_{\mathcal{Y}}^2(E) = 0$ as claimed.

Even if E is not a purely transcendental extension of F , we can still obtain local-global principles if K is a local field. Theorem 5.5 below is a consequence of deep results of Saito-Sato, of de Jong and of Gabber, and of Kerz-Saito. We thank J.-L. Colliot-Thélène for pointing out a theorem of Saito-Sato for the case $n = 3$.

Theorem 5.5. *Let K be a non-archimedean local field, i.e. a complete discretely valued field with finite residue field. Let F be a semi-global field over K , and let E be the function field of a regular projective curve Y over F . Then $\mathbb{III}_F^n(E) = 0, n \geq 3$.*

Proof. Assume first that m is prime. Let \mathcal{Y} be a normal proper model of E over \mathcal{O}_K , the ring of integers of K . Let $\mathcal{Y}' \rightarrow \mathcal{Y}$ be an alteration morphism of degree prime to m , with \mathcal{Y}' proper over \mathcal{O}_K and regular, and with the special fiber a simple normal crossings divisor (see

the uniformization theorem of de Jong [dJ96, Theorem 4.1] and its refinement by Gabber [IT14, Theorem 2.4]; cf. also [Tem17, Theorem 1.2.5]). Let E' be the field of functions of \mathcal{Y}' .

For $n = 3$, since \mathcal{Y}' is a regular scheme of total dimension 3 that is proper over \mathcal{O}_K , with special fiber a simple normal crossings divisor, a theorem of Saito and Sato ([SS10, Theorem 2.13], see also [CT11, Théorème 3.16]) yields the vanishing of the group $H_{\text{nr}}^3(E'/\mathcal{Y}')$. (Here $H_{\text{nr}}^3(E'/\mathcal{Y}') = KH_3(\mathcal{Y}', \mathbb{Z}/m\mathbb{Z})$ in the notation of [SS10], since $\dim \mathcal{Y}' = 3$.) Hence its subgroup $H_{\text{nr}}^3(E')$ also vanishes.

For $n = 4$, since \mathcal{Y}' is a regular scheme that is proper over \mathcal{O}_K , the Kato conjecture (proved by Kerz and Saito, [KS12, Theorem 8.1]) asserts that the group $H_{\text{nr}}^4(E'/\mathcal{Y}')$ is zero. Hence so is its subgroup $H_{\text{nr}}^4(E')$.

For $n \geq 5$, the group $H^n(E')$ is zero by cohomological dimension.

Since $[E' : E]$ is prime to m , the map $H_{\text{nr}}^n(E) \rightarrow H_{\text{nr}}^n(E')$ is injective, and thus $H_{\text{nr}}^n(E) = 0$ for $n \geq 3$. Hence so is $\text{III}_F^n(E)$, which is contained in $H_{\text{nr}}^n(E)$ by Proposition 4.6. This concludes the case when m is prime. The general case follows by the same argument as in the proof of Corollary 4.4 (replacing $\text{III}_F^n(F)$ by $\text{III}_F^n(E)$ in that proof). \square

Remark 5.6. In the particular case that \mathcal{Y} is a smooth conic over F , the vanishing of $\text{III}_F^3(E)$ also follows via [PS16, Theorem 6.4] (which assumed that K is p -adic, but carries over to more general local fields K if F is the function field of a smooth K -curve).

Remark 5.7. Under the hypothesis that $[F(\zeta_m) : F]$ is prime to m , a restriction-corestriction argument shows that the above vanishing results about $\text{III}_F^n(E) = \text{III}_{\Omega_F}^n(E, \mu_m^{\otimes(n-1)})$ also hold for $\text{III}_{\Omega_F}^n(E, \mu_m^{\otimes r})$ for any r . The same is the case for the results in the next subsection about the vanishing of $\text{III}_{F,0}^n(F)$ and $\text{III}_{F,0}^n(E)$.

Remark 5.8. A related situation, over a global field rather than a local field, was considered by Jean-Louis Colliot-Thélène and Bruno Kahn in [CTK13], in the case $n = 3$. Section 8 there introduced a III-type group of unramified classes $\text{III}H_{\text{nr}}^3(X, \mathbb{Q}/\mathbb{Z}(2)) := \ker[H_{\text{nr}}^3(K(X), \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \prod_v H_{\text{nr}}^3(K_v(X_v), \mathbb{Q}/\mathbb{Z}(2))]$, where X is a smooth projective variety over a global field K , and where the product is over all places v of K . Conjecture 8.9 in [CTK13] asserts in particular that the group $\text{III}H_{\text{nr}}^3(X, \mathbb{Q}/\mathbb{Z}(2))$ vanishes if X is a geometrically ruled surface over a function field in one variable over a finite field.

5.2. Local-global principles on the general fiber. The goal of this section is to investigate the case when the chosen set of discrete valuations does not include the valuations centered on the closed fiber. Namely, we consider the set $\Omega_{F,0}$ of discrete valuations that are trivial on K ; i.e., those that correspond to closed points of the generic fiber of \mathcal{X} . We write $\text{III}_{F,0}^n$ for the corresponding III-groups, and in particular we write $\text{III}_{F,0}^n(F) = \text{III}_{\Omega_{F,0}}^n(F, \mu_m^{\otimes(n-1)})$. In the case of local-global principles for semi-global fields, this situation was considered in [CH15], [HSS15], and [HS16].

In this situation, the condition for a class $\alpha \in H^n(E)$ to be trivial over the fields E_v for $v \in \Omega_{F,0}$ is much weaker than it is when we consider the set Ω_F . In particular, if K is a local field, we show in Theorem 5.12 that for $n > 2$ the groups $\text{III}_{F,0}^n$ can be nontrivial even if E is a purely transcendental field extension of F (contrary to what happens in Theorem 5.2, where the larger set of discrete valuations was used).

Lemma 5.9. *Let F be a semi-global field over a non-archimedean local field K . Then $\text{III}_{F,0}^1(F) = \text{III}_F^1(F)$ and $\text{III}_F^n(F) \subseteq \text{III}_{F,0}^n(F)$ for $n > 1$.*

Proof. The containment $\text{III}_F^n(F) \subseteq \text{III}_{F,0}^n(F)$ is immediate for all $n \geq 1$ since $\Omega_{F,0} \subset \Omega_F$. It remains to show that $\text{III}_{F,0}^1(F) \subseteq \text{III}_F^1(F)$.

Pick a regular model \mathcal{X} of F . Then each element $f \in \text{III}_{F,0}^1(F) = \text{III}_{F,0}^1(F, \mathbb{Z}/m\mathbb{Z})$ corresponds to a $\mathbb{Z}/m\mathbb{Z}$ -Galois branched cover $\mathcal{X}' \rightarrow \mathcal{X}$ that is split at every closed point of the generic fiber of \mathcal{X} . We claim that $\mathcal{X}' \rightarrow \mathcal{X}$ is also split over the generic point of each irreducible component of the reduced closed fiber. Once this is shown, by [HHK15, Corollary 5.3] it follows that $\mathcal{X}' \rightarrow \mathcal{X}$ is a split cover; i.e., it is split over every point of \mathcal{X} except possibly the generic point. Then by [HHK15, Proposition 8.2], it follows that $f \in \text{III}_F^1(F) = \text{III}_F^1(F, \mathbb{Z}/m\mathbb{Z})$, which will complete the proof.

To prove the claim, it suffices to show that it holds in the case that m is a prime, since a $\mathbb{Z}/m\mathbb{Z}$ -Galois branched cover is a tower of cyclic covers of prime order, and since a split cover is étale and therefore each step in the tower is also a regular model. So for the remainder of the proof we assume that $m = \ell$, a prime (unequal to the residue characteristic of \mathcal{O}_K).

To show the claim with $m = \ell$, we may assume that $\zeta_\ell \in K$, since $[K(\zeta_\ell) : K]$ has degree prime to ℓ and so it would suffice to show that the base change to $K(\zeta_\ell)$ satisfies the splitness condition. With $\zeta_\ell \in K$, the branched cover $\mathcal{X}' \rightarrow \mathcal{X}$ is Kummer, and so is given generically by adjoining an ℓ -th root of some element $a \in K$.

Let C be an irreducible component of the reduced closed fiber of \mathcal{X} . Pick a regular closed point P of the reduced closed fiber such that P lies on C , and take a regular system of parameters σ, τ for $\mathcal{O}_{\mathcal{X},P}$. Since P is a regular point of the reduced closed fiber, C is the unique irreducible component of the closed fiber passing through P ; and at most one of the two parameters vanishes at the generic point of C . Thus at least one of these two parameters, say σ , instead vanishes along the closure D in $\text{Spec}(\mathcal{O}_{\mathcal{X},P})$ of a closed point z of the general fiber X of \mathcal{X} . Since $\mathcal{X}' \rightarrow \mathcal{X}$ is split over every point of the general fiber, it follows that $a = \tau^r u$, where u is a unit in $\mathcal{O}_{\mathcal{X},P}$ and where we may choose a so that $0 \leq r < \ell$.

Now D is the spectrum of a complete discrete valuation ring whose uniformizer is the reduction of τ modulo σ . By hypothesis, $\mathcal{X}' \rightarrow \mathcal{X}$ splits over z . Since the extension of K corresponding to $\mathcal{X}' \rightarrow \mathcal{X}$ is obtained by adjoining an ℓ -th root of a , it follows that $r = 0$. Thus $\mathcal{X}' \rightarrow \mathcal{X}$ is étale over $\text{Spec}(\mathcal{O}_{\mathcal{X},P})$, and hence is unramified over the generic point of C . Moreover $\mathcal{X}' \rightarrow \mathcal{X}$ is split over P , since its restriction over $\text{Spec}(D) \subset \text{Spec}(\mathcal{O}_{\mathcal{X},P}) \subset \mathcal{X}$ is an étale cover that is split over its generic point z . Hence the restriction of $\mathcal{X}' \rightarrow \mathcal{X}$ to C is a branched cover that is split over all but possibly finitely many closed points (viz., over all the closed points of C where the closed fiber is regular). Since C is a curve over a finite field, Chebotarev's density theorem implies that this branched cover of C is trivial. This shows that $\mathcal{X}' \rightarrow \mathcal{X}$ splits over the generic point of every irreducible component of the closed fiber of \mathcal{X} , proving the claim. \square

Lemma 5.10. *Let F be the function field of a smooth projective curve X over a non-archimedean local field K of residue characteristic p . Let \mathcal{X} be a regular proper model of X over the ring of integers of K and let Γ be the reduction graph associated to the closed fiber of \mathcal{X} . Then*

- (a) $\text{III}_{F,0}^1(F) = \text{Hom}(\pi_1(\Gamma), \mathbb{Z}/m\mathbb{Z})$, and this group is trivial if and only if Γ is a tree.
- (b) the groups $\text{III}_{F,0}^1(F)$ and $\text{III}_{F,0}^3(F)$ are dual; i.e., there is a perfect pairing to \mathbb{Q}/\mathbb{Z} .

(c) $\text{III}_{F,0}^2(F) = 0$.

Proof. First we prove (a). By Lemma 5.9, $\text{III}_{F,0}^1(F) = \text{III}_F^1(F)$. The elements of $\text{III}_F^1(F)$ define $\mathbb{Z}/m\mathbb{Z}$ -Galois split covers of \mathcal{X} ; and conversely, each $\mathbb{Z}/m\mathbb{Z}$ -Galois split cover of \mathcal{X} corresponds to an element $H^1(F, \mathbb{Z}/m\mathbb{Z})$ that lies in $\text{III}_F^1(F)$, by [HHK15, Proposition 8.2]. By [HHK15, Corollary 6.4], the set of split covers of \mathcal{X} is classified by $\text{Hom}(\pi_1(\Gamma), \mathbb{Z}/m\mathbb{Z})$. So assertion (a) follows.

For part (b), the case where k is a p -adic field follows from [HS16, Theorem 4.4], which asserts in this situation that for any finite Galois module A there is a perfect pairing between $\text{III}_{F,0}^i(F, A)$ and $\text{III}_{F,0}^{4-i}(F, A(-2))$; viz., we can take $i = 3$ and $A = \mu_m^{\otimes 2}$. The proof of that theorem carries over mutatis mutandis to the case of a smooth curve over an equal characteristic p local field, provided that the torsion in A is prime to p . Since our choice of m is assumed prime to the residue characteristic p , part (b) follows in that case too.

For (c), first note that $\text{III}_{F,0}^2(F) = \text{III}_{F,0}^2(F, \mu_m)$ consists of the m -torsion elements of $\text{Br}(F)$ that are locally trivial with respect to $\Omega_{F,0}$. Thus if α is a non-trivial element of $\text{III}_{F,0}^2(F, \mu_m)$, then a multiple of α is a non-trivial element of $\text{III}_{F,0}^2(F, \mu_\ell)$, for some prime ℓ that divides m . Hence we are reduced to proving the result in the case that $m = \ell$ is prime. In that situation, $[F[\zeta_m] : F] = [K[\zeta_m] : K]$ is prime to m ; and so by a restriction-corestriction argument we may assume that ζ_m lies in K , and hence in \mathcal{O}_K .

Now let $\alpha \in \text{III}_{F,0}^2(F) = \text{III}_{F,0}^2(F, \mu_m)$. We claim that $\partial_s(\alpha) = 0$ for every codimension 1 point $s \in \mathcal{X}^{(1)}$. By [Gro68, Théorème 3.1, p. 98], the Brauer group of a regular curve \mathcal{X} over the ring of integers of K is equal to that of its closed fiber and hence is trivial. Thus the claim will imply that $\alpha = 0$, and so will give (c).

Since the residue map at any closed point $x \in X$ factors through F_v , we deduce that α is unramified at x . Hence it only remains to check the claim that $\partial_s(\alpha) \in H^1(\kappa(s), \mathbb{Z}/m\mathbb{Z})$ is trivial for s a generic point of a component C of the reduced closed fiber of \mathcal{X} . But $\zeta_m \in \kappa(s)$, since $\zeta_m \in \mathcal{O}_K$. So we may identify $H^1(\kappa(s), \mathbb{Z}/m\mathbb{Z})$ with $H^1(\kappa(s), \mu_m) = \kappa(s)^\times / \kappa(s)^{\times m}$, and identify $\partial_s(\alpha)$ with the m -th power class $[f]$ of some $f \in \kappa(s)^\times$.

Let $P \in C$ be a closed point in the regular locus of the reduced closed fiber. The local ring $\mathcal{O}_{\mathcal{X},P}$ is a regular local ring with maximal ideal generated by two local parameters. The zero locus of one of these parameters, which we call u , defines a regular connected closed subscheme $Z \subset \mathcal{X}$ that is not contained in the closed fiber. Let z be the generic point of Z . Consider the element $\beta = \alpha \cdot (u)$. Since α is trivial in F_z we deduce that β is unramified at Z . Hence β can be ramified on \mathcal{X} only along the closed fiber. Then from the Gersten complex we obtain that

$$0 = \partial_P \partial_z(\alpha \cdot (u)) = -\partial_P \partial_s(\alpha \cdot (u)) = -\partial_P((f) \cdot (\bar{u})),$$

where \bar{u} is the class of u modulo the ideal of the reduced closed fiber. Hence, $\partial_P((f) \cdot (\bar{u})) = 0$. Similarly, we have that $\partial_P(f) = \partial_P \partial_z(\alpha) = 0$, so that f comes from $\widehat{\mathcal{O}}_{C,P}^\times$. But since $\partial_P((f) \cdot (\bar{u}))$ is trivial, we deduce that the image of f in the completion of $\kappa(C)$ at P is an m -th power, for any $P \in C$ in the regular locus of the reduced closed fiber. By Chebotarev density, we deduce that f is itself an m -power. Thus $\partial_s(\alpha) = 0$, proving the claim and hence (c). \square

Remark 5.11. (a) Part (c) of the above lemma was previously shown in the p -adic case by using Lichtenbaum's duality pairing, in the second paragraph of the proof of part (2) of [HS16, Proposition 3.4].

- (b) In the above lemma, the assumption of smoothness (and not just regularity) was used only in part (b). There, it was needed in order to carry over to the equal characteristic case the proof of [HS16, Theorem 4.4], which relies on Artin-Verdier duality and hence on Poincaré duality. As a result, smoothness is also used in the parts of the next theorem that rely on Lemma 5.10(b); viz., the case $n = 4$ of part (a), and part (b).

Theorem 5.12. *Let F be the function field of a smooth projective curve over a non-archimedean local field K of residue characteristic p and let $E = F(\mathbb{P}_F^1)$ be a purely transcendental field extension of F . Then*

- (a) $\mathbb{III}_{F,0}^n(E) \neq 0$ for $n = 2, 4$.
(b) The group $\mathbb{III}_{F,0}^3(E)$ is trivial if and only if the reduction graph of a regular model \mathcal{X} of F over the ring of integers of K is a tree.

Proof. We consider the exact sequence in Proposition 3.2 with $\Omega = \Omega_{F,0}$.

For part (a), take $n = 2$ in this exact sequence. By Lemma 5.1, there is a finite separable extension F' of F having a model whose reduction graph is not a tree. There is a point $y \in \mathbb{A}_F^{1(1)}$ with $\kappa(y) = F'$; and the corresponding term in the right hand side of the exact sequence is non-zero by Lemma 5.10(a). (Here we use that the valuations in $\Omega_{F',0}$ are precisely the extensions to F' of the valuations in $\Omega_{F,0}$.) Hence the middle term of the exact sequence, $\mathbb{III}_{F,0}^2(E)$, is also non-zero, proving the case $n = 2$.

Similarly, taking $n = 4$ in this exact sequence, Lemma 5.10(b) together with the previous case imply that the right hand term in the sequence is non-zero. Hence $\mathbb{III}_{F,0}^4(E)$ is nonzero as well.

For part (b), take $n = 3$ in this exact sequence. By Lemma 5.10(c), the right hand term in the sequence vanishes, and so the map $\mathbb{III}_{F,0}^3(F) \rightarrow \mathbb{III}_{F,0}^3(E)$ is an isomorphism. Hence the assertion follows from Lemma 5.10(a),(b). \square

We note that the case $n = 2$ above can also be deduced from Theorem 5.2(b) together with the fact that $\mathbb{III}_F^2(E) \subseteq \mathbb{III}_{F,0}^2(E)$, where this inclusion follows from the containment $\Omega_{F,0} \subset \Omega_F$.

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