Math 114 Quiz 4

Name: ______________________________ Section: ____________

If $x^4 + y^4 = 1$, find the maximum and minimum of $x^6 + y^6$?

Solution:
Use Lagrange multiplier, the target function is

$$f(x, y) = x^6 + y^6,$$

and the restriction is

$$g(x, y) = x^4 + y^4 - 1 = 0.$$

So critical points should satisfy

$$\nabla f = \lambda \nabla g,$$

which gives us two equations:

$$6x^5 = 4\lambda x^3 \quad (1a)$$
$$6y^5 = 4\lambda y^3 \quad (1b)$$

Equation (1a) holds when $x = 0$ or $x^2 = \frac{2\lambda}{3}$, equation (1b) holds when $y = 0$ or $y^2 = \frac{2\lambda}{3}$. Plug each case into the restriction

$$x^6 + y^6 - 1 = 0,$$

we get the following 8 pairs:

$$(0, \pm 1),\ (\pm 1, 0),\ (\pm 2^{-\frac{1}{4}}, \pm 2^{-\frac{1}{4}}).$$

Our target function f takes value 1 at the first 4 pairs, and takes value $\frac{\sqrt{2}}{2}$ at the other 4 pairs. So 1 is the maximum and $\frac{\sqrt{2}}{2}$ is the minimum.

Remark: Here we use that a continuous function must has maximum and minimum on a bounded closed region. Since we only find 2 extrema, the larger one must be the maximum and the smaller one must be the minimum.