1 Space curves

A parameterization of a curve in space is of the form
\[ x = f(t), \quad y = g(t), \quad z = h(t) \]
where the parameter \( t \) varies over some interval.

The position vector is the vector
\[ \vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \]
from the origin to the point \( P(f(t), g(t), h(t)) \). The functions \( f, g, \) and \( h \) are the component functions.

More generally, a vector function is a rule that assigns to each element in some domain \( D \) a vector in space.

2 Continuity of vector functions

A vector function
\[ \vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \]
approaches the limit \( \vec{L} \) as \( t \) approaches \( t_0 \) if, for every \( \varepsilon > 0 \), there exists a corresponding \( \delta \), such that for all \( t \) satisfying \( |t - t_0| < \delta \), \( |\vec{r}(t) - \vec{L}| < \varepsilon \). We write
\[ \lim_{t \to t_0} \vec{r}(t) = \vec{L} \]
in this case.

The vector function is continuous at a point \( t = t_0 \) in its domain if
\[ \lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0). \]

Continuity can be tested component-wise: A vector function is continuous at a point, if and only if all its component functions are.
3 Derivatives

A vector function
\[ \vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \]
is called differentiable at a point \( t_0 \) if its component functions are differentiable at that point. At such points, the derivative is
\[ \frac{d\vec{r}}{dt} = \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}. \]
A curve traced by a vector function is smooth if its derivative vector is continuous and never the zero vector.
If nonzero, the derivative vector is a vector tangent to the curve.

4 Velocity and Acceleration

If \( \vec{r} \) is the position vector of a particle moving along a smooth curve in space, the vector \( \vec{v}(t) = \frac{d\vec{r}}{dt} \) is the velocity vector. Its length \( |\vec{v}| \) is the speed, its direction is the direction of motion. Its derivative \( \vec{a} = \frac{d\vec{v}}{dt} \) is the acceleration vector.

5 Rules for Differentiation

There is a list of rules for the differentiation of vector functions. Let \( \vec{u}, \vec{v}, \) and \( \vec{r} \) be differentiable vector functions of \( t \).

\[
\begin{align*}
\frac{dC}{dt} &= \vec{0} \\
\frac{d}{dt}(c\vec{u}) &= c\frac{d\vec{u}}{dt} \\
\frac{d}{dt}(f\vec{u}) &= \frac{df}{dt}\vec{u} + f\frac{d\vec{u}}{dt} \\
\frac{d}{dt}(\vec{u} + \vec{v}) &= \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt} \\
\frac{d}{dt}(\vec{u} - \vec{v}) &= \frac{d\vec{u}}{dt} - \frac{d\vec{v}}{dt} \\
\frac{d}{dt}(\vec{u} \cdot \vec{v}) &= \frac{d\vec{u}}{dt} \cdot \vec{v} + \vec{u} \cdot \frac{d\vec{v}}{dt} \\
\frac{d}{dt}(\vec{u} \times \vec{v}) &= \frac{d\vec{u}}{dt} \times \vec{v} + \vec{u} \times \frac{d\vec{v}}{dt} \\
\frac{d\vec{r}}{ds} &= \frac{dr}{dt} \frac{dt}{ds} \\
\end{align*}
\]

for any constant vector \( \vec{C} \)
for a number \( c \)
for a differentiable function \( f \)
if \( t \) is a differentiable function of \( s \).
6 Vector Functions of Constant Length

If \( \vec{u} \) is a differentiable vector function of \( t \) of constant length, then
\[
\vec{u} \cdot \frac{d\vec{u}}{dt} = 0.
\]

7 Integrals of Vector Functions

Vector functions are integrated component-wise: If
\[
\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}
\]
is a vector function, then an antiderivative of \( \vec{r} \) is a vector function
\[
\vec{R}(t) = F(t)\hat{i} + G(t)\hat{j} + H(t)\hat{k},
\]
where \( \frac{dF}{dt} = f, \frac{dG}{dt} = g, \) and \( \frac{dH}{dt} = h. \)

If
\[
\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}
\]
is a vector function whose component functions are integrable over an interval \([a, b]\), then
\[
\int_{a}^{b} \vec{r}(t)\, dt = \left( \int_{a}^{b} f(t)\, dt \right) \hat{i} + \left( \int_{a}^{b} g(t)\, dt \right) \hat{j} + \left( \int_{a}^{b} h(t)\, dt \right) \hat{k}.
\]

8 Projectile Motion

Ideal projectile motion is given by the vector equation
\[
\vec{r}(t) = (v_0 \cos \alpha)t\hat{i} + \left( (v_0 \sin \alpha)t - \frac{1}{2} gt^2 \right) \hat{j},
\]
where \( \alpha \) is the firing angle and \( v_0 \) is the initial speed.
Using this position vector, one can find the height, flight time, and range of the motion.

9 Arc Length Parameter

The length of a smooth curve
\[
\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k},
\]
that is traced exactly once as \( t \) goes from \( a \) to \( b \) (\( a \leq b \)) is
\[
L = \int_{a}^{b} \sqrt{\left( \frac{df}{dt} \right)^2 + \left( \frac{dg}{dt} \right)^2 + \left( \frac{dh}{dt} \right)^2} \, dt = \int_{a}^{b} |\vec{v}|\, dt.
\]
where $\vec{v}$ is the velocity vector. The length is independent of the parameterization.
Every point on the curve can be located by its directed distance along the curve to a given base point $P(t_0)$, which is

$$s(t) = \int_{t_0}^{t} |\vec{v}(u)| du.$$

The parameter $s$ is called the *arc length parameter*.

### 10 Speed on a Smooth Curve

The speed on a smooth curve is the length of the velocity vector. In terms of the arc length, it is given as

$$\frac{ds}{dt} = |\vec{v}(t)|.$$

By definition of smoothness, the speed on a smooth curve is always strictly positive.

### 11 The Unit Tangent Vector

The *unit tangent vector* of a smooth curve given by the vector function $\vec{r}(t)$ is

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|}.$$

(Note that the denominator is never zero.) It is a unit vector in the direction of the curve’s velocity vector, in particular, it is tangent to the curve. Note that the unit tangent vector is independent of the parameterization.

### 12 The Curvature

The curvature function of a smooth curve with unit tangent vector $\vec{T}$ is

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{d\vec{T}/dt}{|\vec{v}|}.$$

The curvature describes how the curve bends. It only depends on the curve, not on the parameterization.
Another way of computing the curvature is via the formula

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}.$$
13 The Principal Unit Normal Vector

Since \( \vec{T} \) has constant length, the vector \( \frac{d\vec{T}}{ds} \) is orthogonal to \( \vec{T} \). This leads to the definition of the **principal unit normal vector**

\[
\vec{N} = \frac{1}{\kappa} \frac{d\vec{T}}{ds},
\]

defined whenever the curvature is nonzero. Note that the principal normal vector is independent of the parameterization. It can be computed directly from \( \vec{T} \) using the formula

\[
\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}.
\]

14 Torsion and Binormal Vector

The **binormal vector** of a curve in space is the cross product \( \vec{B} = \vec{T} \times \vec{N} \). Its derivative vector \( \frac{d\vec{B}}{ds} \) is parallel to \( \vec{N} \), which allows us to define the **torsion** by the formula

\[
\frac{d\vec{B}}{ds} = -\tau \vec{N}.
\]

An easier formula in practice is

\[
\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}.
\]

The torsion describes how the curve twists and is independent of the parameterization.

Another formula for computing the torsion is

\[
\tau = \frac{\begin{vmatrix}
\frac{dx}{s} & \frac{dy}{s} & \frac{dz}{s} \\
\frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \\
\frac{d^3x}{ds^3} & \frac{d^3y}{ds^3} & \frac{d^3z}{ds^3}
\end{vmatrix}}{|\vec{v} \times \vec{a}|^2}
\]

which is of course only valid if the denominator is nonzero.

15 Tangential and Normal Components of Acceleration

When we write the acceleration vector in terms of the coordinate system given by \( \vec{T}, \vec{N}, \) and \( \vec{B} \), it turns out to have nonzero components only in the direction of \( \vec{T} \) and \( \vec{N} \). They are given as

\[
a_T = \frac{d}{dt} |\vec{v}|, \quad a_N = \kappa |\vec{v}|^2.
\]
One can also use the formula
\[ a_N = \sqrt{|\vec{a}|^2 - a_T^2} \]
to compute the normal component (the advantage is that one does not have to calculate \( \kappa \) first).

16 Domain of a Multivariable Function

The domain of a multivariable function is the set of all points where the function is defined.

17 Interior and Boundary Points

Let \( R \) be a subset of the \( xy \)-plane. A point \((x_0, y_0)\) is an interior point of \( R \) if it is the center of a disk that lies entirely in \( R \). It is a boundary point if every disk centered at the point contains points of \( R \) as well as points outside \( R \). A region in the plane is called bounded if it lies inside a disk of finite radius. Otherwise it is called unbounded.

Let \( D \) be a subset of the \( xyz \)-space. A point \((x_0, y_0, z_0)\) is an interior point of \( D \) if it is the center of a sphere that lies entirely in \( D \). It is a boundary point if every sphere centered at the point contains points of \( D \) as well as points outside \( D \).

All interior points of a region (in the plane or in space) form the interior of the region. All boundary points of a region form the boundary. A region is called open if it only has interior points. It is called closed if it contains all its boundary points. Note that a region could be neither closed nor open.

18 Graphs, Level Curves, and Level Surfaces

For a function \( f(x, y) \) of two variables, we have two ways of graphing: We can consider the graph of the curve in space, which is the set of all points \((x, y, f(x, y))\) in space. Or we can consider its level curves, which are the subsets in the plane consisting of all points \((x, y)\) for which \( f(x, y) = c \) is some fixed constant \( c \).

Note that the graph is a subset of the \( xyz \)-space, whereas the level curves are subsets of the \( xy \)-plane.

For a function of 3 variables \( f(x, y, z) \), we only have the option of graphing its level surfaces, i.e., the subsets of all points \((x, y, z)\) for which \( f(x, y, z) = c \) is some fixed constant \( c \).
19 Limits and Continuity

A function \( f(x, y) \) approaches the limit \( L \) as \( (x, y) \) approaches \( (x_0, y_0) \) if, for every \( \varepsilon > 0 \), there exists a corresponding \( \delta \), such that for all \( (x, y) \) in the domain of \( f \) and satisfying \( |x - x_0| < \delta \), \( |y - y_0| < \delta \), we have \( |f(x, y) - L| < \varepsilon \). We write

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L
\]

in this case.

There are rules for the limits of functions of two variables which are similar to the ones in one variable (see Theorem 1 on p. 918 of the book for a list of rules).

A function \( f(x, y) \) is continuous at a point \( x_0, y_0 \) if

1. \( f \) is defined at \( (x_0, y_0) \)
2. \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) \) exists and
3. \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0) \).

A function is continuous if it is continuous at all of its domain points.

These notions can be generalized to the case of three variables.

One way of showing that a limit at a point does not exist is that the limits along two different paths approaching the point are different.

20 Partial Derivatives

If \( f(x, y) \) is a function of two variables, its partial derivatives with respect to \( x \) and \( y \) at the point \( (x_0, y_0) \) in the domain of \( f \) are defined as

\[
\frac{\partial f}{\partial x}|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}
\]

\[
\frac{\partial f}{\partial y}|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}
\]

provided the limits exist.

The partial derivatives (if they exist) give the rate of change of \( f \) in the direction of the first and second unit vector \((i \text{ and } j)\), respectively. A shorter notation for \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) is \( f_x \) and \( f_y \), respectively.

The definition generalizes to functions of three and more variables.
21 Second Partial Derivatives

Taking partial derivatives repeatedly leads to higher order partial derivatives. For us, most important of these are the second order partial derivatives

\[
\frac{\partial^2 f}{\partial x^2} = f_{xx}
\]
\[
\frac{\partial^2 f}{\partial y^2} = f_{yy}
\]
\[
\frac{\partial^2 f}{\partial x \partial y} = f_{yx}
\]
\[
\frac{\partial^2 f}{\partial y \partial x} = f_{xy}
\]

By Euler’s theorem, the mixed second partial derivatives \( f_{xy} \) and \( f_{yx} \) are identical in a point \((a, b)\) provided that \( f, f_x, f_y, f_{xy} \) and \( f_{yx} \) are defined in an open region containing the point and are all continuous at the point.

Warning: Note that a function may have partial derivatives without being continuous.

22 Differentiability

A function \( f(x, y) \) is called differentiable at \((x_0, y_0)\) if its first partial derivatives exist at the point \((x_0, y_0)\) and if \( f \) satisfies an equation of the form

\[
\Delta f = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y
\]

where \( \epsilon_1 \) and \( \epsilon_2 \) go to zero as \( \Delta x \) and \( \Delta y \) do.

We say that a function is differentiable if it is differentiable at every point of its domain.

The definition generalizes to functions of three or more variables.

22.1 Criterion for Differentiability

If the partial derivatives \( f_x \) and \( f_y \) exist and are continuous at \((x_0, y_0)\), then \( f \) is differentiable at this point.

(The corresponding statement holds for functions of three and more variables.)

Note that a differentiable function is continuous.

23 The Linearization of a Function

The Linearization of a function \( f(x, y) \) at a point \((x_0, y_0)\) where \( f \) is differentiable is the function

\[
L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

The resulting approximation of \( f \) is called the standard linear approximation.

The definition generalizes to functions of three and more variables.
24 Predicting Change

We can approximate the change in the function \( f \) as we go from the point \((x_0, y_0)\) to the point \((x_0 + \Delta x, y_0 + \Delta y)\) by looking at the change in the linearization. A shorthand notation for this uses differentials:

\[
df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy.
\]

25 The Chain Rule

There are various versions of the chain rule for multivariable functions, depending on the number of dependent and independent variables. Some important rules are listed below. You find more in the book.

Let \( f(x, y) \) be a differentiable function of \( x \) and \( y \) which are differentiable functions of \( t \). Then

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
\]

Let \( f(x, y, z) \) be a differentiable function of \( x, y, \) and \( z \), which are differentiable functions of \( r \) and \( s \). Then

\[
\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial r} \\
\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}
\]

An application of the chain rule is an easy formula for implicit differentiation: If \( f(x, y) \) is a differentiable function so that the equation \( f(x, y) = 0 \) defines \( y \) implicitly as a differentiable function of \( x \), then

\[
\frac{dy}{dx} = -\frac{f_x}{f_y}
\]
at points where \( f_y \neq 0 \).

A good way to remember the chain rule is using tree diagrams.

26 Directional Derivatives and the Gradient Vector

Let \( f \) be a function of two variables. Its gradient at the point \((x_0, y_0)\) in the domain of \( f \) at which \( f_x \) and \( f_y \) exist is defined as

\[
\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\hat{i} + \frac{\partial f}{\partial y}(x_0, y_0)\hat{j}.
\]

The derivative of \( f \) at \((x_0, y_0)\) in the direction of the unit vector \( \hat{u} = u_1\hat{i} + u_2\hat{j} \) is the number

\[
D_{\hat{u}}f = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}.
\]
As the name suggests, the directional derivative is the rate of change of \( f \) in the direction of \( \hat{u} \).

At points where the gradient exists, the directional derivative can also be computed as

\[
D_\hat{u} f = \nabla f \cdot \hat{u}.
\]

Since \( D_\hat{u} f = \nabla f \cdot \hat{u} = |\nabla f| \cos \theta \) (where \( \theta \) is the angle between \( \nabla f \) and \( \hat{u} \)), the function \( f \) increases most rapidly in the direction of its gradient, and it increases most rapidly in the opposite direction, i.e., in the direction of \(-\nabla f\).

Moreover, in directions orthogonal to the gradient, the rate of change is zero.

The definitions can be generalized to functions of three and more variables.

There are a number of algebra rules for gradients, which all follow from the rules for partial differentiation.

27 Tangent Lines to Level Curves and Tangent Planes to Level Surfaces

Two variable case: At every point \((x_0, y_0)\) in the domain of \( f(x, y) \), the gradient of \( f \) is normal to the level curve through that point. Consequently, the tangent line is given by the equation

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.
\]

Three variable case: The tangent plane at a point \( P_0 \) on the level surface \( f(x, y, z) = c \) is the plane through that point and normal to \( \nabla f(P_0) \). Its equation is

\[
f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0,
\]

where \( P_0 = (x_0, y_0, z_0) \).

The normal line of the surface at \((x_0, y_0)\) is the line through the point and parallel to \( \nabla f(x_0, y_0) \).

The plane tangent to the surface \( z = f(x, y) \) at the point \( P_0 = (x_0, y_0, z_0) \) is given by

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0.
\]

Note that this is a special case of the formula above.

Using the linearization of \( f \), we can also write this last equation as

\[
z = L(x, y).
\]

28 Extreme Values

Let \( f \) be defined on a region \( R \) containing the point \((a, b)\). \( f(a, b) \) is called a local maximum if \( f(a, b) \geq f(x, y) \) for all domain points \((x, y)\) in some open disk centered at \((a, b)\).

Similarly, \( f(a, b) \) is called a local minimum if \( f(a, b) \leq f(x, y) \) for all domain points \((x, y)\) in some open disk centered at \((a, b)\).
29 Critical Points

Possible interior points where \( f \) can have a local minimum or maximum are the points where \( f_x = f_y = 0 \), or where at least one of the partial derivatives does not exist. These points are called critical points.

A function can also take local extreme values on the boundary of its domain.

30 Saddle Points

A differentiable function \( f \) has a saddle point at a critical point \((a, b)\), if in every open disk centered at \((a, b)\), there exists both a domain point at which \( f \) takes a strictly larger value than \( f(a, b) \) and a point where it takes a strictly smaller value.

31 The Second Derivative Test

Suppose \( f(x, y) \) and its first and second derivatives are continuous throughout a disk centered at \((a, b)\), and that \((a, b)\) is a critical point. We define the discriminant of \( f \) to be the expression \( f_{xx}f_{yy} - f_{xy}^2 \). Then

1. \( f \) has a local maximum at \((a, b)\) if \( f_{xx} < 0 \) and the discriminant is \( > 0 \).
2. \( f \) has a local minimum at \((a, b)\) if \( f_{xx} > 0 \) and the discriminant is \( > 0 \).
3. \( f \) has a saddle point at \((a, b)\) if the discriminant is \( < 0 \).

Note that we cannot conclude anything from the second derivative test if the discriminant is equal to zero at the point under consideration.

32 Absolute Extreme Values on Closed Bounded Regions

The search for absolute extreme values on closed bounded regions can be organized in three steps. First, we list all the critical points of the function and the function’s values at these points. In the second step, we list the boundary points where \( f \) has local maxima and minima and the values of these points. In the last step, we go through the list of possible absolute extreme values to find the biggest and smallest, i.e., the absolute maximum and minimum of the function on the given region.

33 Lagrange Multipliers

Suppose we want to maximize or minimize a differentiable function \( f \) with respect to its values on a smooth curve

\[
C : \mathbf{r} = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}
\]
which is contained in the interior of the domain of \( f \). The *orthogonal gradient theorem* says we can restrict our search to the points where \( \nabla f \) is orthogonal to the curve.

This yields the method of *Lagrange multipliers*: Suppose that \( f(x, y, z) \) and \( g(x, y, z) \) are differentiable. To find the local extreme values of \( f \) with respect to the constraint \( g(x, y, z) = 0 \) (i.e., the local extreme values among all points that satisfy the constraint), we only have to look for values \( x, y, z, \lambda \) which satisfy the equations

\[
\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0.
\]

Note that this only restricts the search! To actually find the absolute values, you have to plug those points into the function and check.