Divisibility of function field class numbers

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Outline

Basic question

- Let $E/\mathbb{F}$ be an elliptic curve over a finite field.
- Let ℓ be a prime number.
- What's the chance that $\ell | \#E(\mathbb{F})$?
- Actually, this was answered by Lenstra ($\mathbb{F} = \mathbb{F}_p$) and Howe ($\mathbb{F} = \mathbb{F}_q$).
- In this talk, we'll give an answer for:
  - Abelian varieties of arbitrary dimension
  - Jacobians of curves of arbitrary genus

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  - Jacobians of curves of arbitrary genus

Sneak preview

- Roughly, we show: The answer is approximately $\frac{1}{\ell}$.
- *Actually, it's closer to* $\frac{1}{\ell-1}$
- Let $\alpha(g,r)$ be the chance that a curve C of genus g satisfies $\mathrm{Jac}(C)(\mathbb{F})[\ell] \cong (\mathbb{Z}/\ell)^r$.

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Sneak preview

g	r	$\alpha(g,r)$
1	0	$\frac{\ell^2-\ell-1}{\ell^2-1}$
1	1	$\frac{1}{\ell}$
1	2	$\frac{1}{\ell(\ell^2-1)}$

Sneak preview

g	r	$\alpha(g,r)$
2	0	$\frac{\ell^6-\ell^5-\ell^4+\ell+1}{(\ell^2-1)(\ell^4-1)}$
2	1	$\frac{\ell^3-\ell-1}{\ell^2(\ell^2-1)}$
2	2	$\frac{\ell^3-\ell-1}{\ell^2(\ell^2-1)^2}$
2	3	$\frac{1}{(\ell^2-1)\ell^4}$
2	4	$\frac{1}{\ell^4(\ell^2-1)(\ell^4-1)}$

Sneak preview

g	r	$\alpha(g,r)$
3	0	$\frac{\ell^{12}-\ell^{11}-\ell^{10}+\ell^7+\ell^5+\ell^4-\ell^3-\ell-1}{(\ell^2-1)(\ell^4-1)(\ell^6-1)}$
3	1	$\frac{\ell^8-\ell^6+\ell^2-\ell^5+\ell-\ell^4+1}{\ell^3(\ell^2-1)(\ell^4-1)}$
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3	5	$\frac{1}{(\ell^2-1)(\ell^4-1)\ell^9}$
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Sample spaces: example

$$\begin{tikzcd} \mathcal{E} \arrow[d] & y^2 = x(x-1)(x-\lambda) \arrow[d] \\ S = \mathbb{P}^1 - \{0, 1, \infty\} & \lambda\text{-line} \end{tikzcd}$$

- Fiber over $\lambda = \lambda_0$ is

$$\mathcal{E}_{\lambda_0} : y^2 = x(x-1)(x-\lambda_0).$$

- Varying choice of $\lambda_0 \in S(\mathbb{F})$ varies elliptic curve $\mathcal{E}_{\lambda_0}/\mathbb{F}$.

Basic question: abelian varieties

- Let $X \to S$ be an abelian scheme over a finite field. What is
$$\frac{|\{s \in S(\mathbb{F}) : \ell \mid |X_s(\mathbb{F})|\}|}{|S(\mathbb{F})|}?$$
- Especially, $\mathcal{C} \to S$ a relative curve. What is
$$\frac{|\{s \in S(\mathbb{F}) : \ell \mid |\mathrm{Jac}(\mathcal{C}_s)(\mathbb{F})|\}|}{|S(\mathbb{F})|}?$$

Think of this as a parametrized family of abelian varieties.

Jacobians and class groups

- Let $C/\mathbb{F}$ be a proper, smooth curve of genus $g \geq 1$.
- Jacobian $\mathrm{Jac}(C)$ is a g-dimensional abelian variety which is:
  - the smallest group variety containing C.
  - the variety parametrizing degree zero line bundles on C.
  - the variety such that $\mathrm{Jac}(C)(\mathbb{F})$ is the ideal class group of $\mathbb{F}(C)$.
- For $g = 1$, canonical isomorphism $\mathrm{Jac}(E) \cong E$.

Applications

- *Algorithm for producing papers:*
  1. identify algorithm or protocol over $\mathbb{Z}/N$.
  2. realize it only uses addition *or* multiplication.
  3. replace $\mathbb{Z}/N$ or $(\mathbb{Z}/N)^\times$ with $\mathrm{Jac}(C)(\mathbb{F})$.
- Some algorithms involve many choices of curve.
- Quantities like

$$\textit{how often does } \ell \mid \#\mathrm{Jac}(C)(\mathbb{F})$$

important for security/run-time analysis

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important for security/run-time analysis

Motivation: Cohen-Lenstra heuristics

- Cohen and Lenstra (1983) conjecturally describe asymptotics of class groups of quadratic imaginary number fields.
- Concretely, a (finite abelian) group H occurs in such class groups with frequency $|\mathrm{Aut}(H)|^{-1}$.
- In particular, $\ell | \mathrm{Cl}(\mathcal{O}_K)$ with frequency

$$\frac{1}{|\mathrm{Aut}(\mathbb{Z}/\ell)|} = \frac{1}{\ell - 1}.$$

Function-field Cohen-Lenstra

- Replace quadratic imaginary number field with quadratic *function field*.
- Consider ideal class groups of fields ($f(x) \in \mathbb{F}[x]$)

$$K_{2,f} = \mathbb{F}(x)[y]/(y^2 - f(x)) \cong \mathbb{F}(x)[\sqrt{f(x)}].$$

- Equivalently, study hyperelliptic Jacobians.
- Friedman and Washington 1989 conjecture: An abelian ℓ-group H occurs as $\mathrm{Cl}(K)[\ell]$ with frequency proportional to $1/|\mathrm{Aut}(H)|$. (They take a limit over all g.)

This conjecture is now a theorem (A.-)

Previous work

Consider $K_{d,f} = \mathbb{F}_q(X)[Y^d - f(X)]$, $f(X) \in \mathbb{F}_q[X]$ monic and separable, $\deg f = n$.

- Friesen 2000 gathers data, bounds, for $(d, n) = (2, 4)$.
- Cardon and Murty 2001: Fix n odd. The number of f with $\ell | h(K_{2,f})$ is at least

$$q^{n(1/2+1/\ell)}.$$

Gives infinite supply of such function fields, but the number of such f is $\sim q^n$, thus

$$\lim_{q\to\infty} \frac{q^{n(1/2+1/\ell)}}{\#K_{2,f}/\mathbb{F}_q} = 0.$$

Previous work

Consider $K_{d,f} = \mathbb{F}_q(X)[Y^d - f(X)]$, $f(X) \in \mathbb{F}_q[X]$ monic and separable, $\deg f = n$.

- Friesen 2000 gathers data, bounds, for $(d, n) = (2, 4)$.
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$$q^{n(1/2+1/\ell)}.$$

- Chakraborty and Mukhopadhyay 2004: n even. They produce at least $q^{n/\ell}/\ell^2$ f with $\ell | h(K_{2,f})$.
- Lee and Pacelli [2004-2006]: (Dedekind rings in) higher degree extensions of $\mathbb{F}(x)$, e.g., $K_{d,f}$.

Previous work – elliptic curves

- Lenstra 1987: the proportion of elliptic curves $E/\mathbb{F}_p$ with $\ell \big| \big|E(\mathbb{F}_p)\big|$ is about $1/(\ell-1)$, if $p \equiv 1 \bmod \ell$.
- Howe 1993: Generalizes to $\mathbb{F}_q$, computes proportion of $E/\mathbb{F}_q$ with given group structure.
- Gekeler 2003: Computes number of elliptic curves over $\mathbb{F}_p$ with given trace and determinant of Frobenius.

Results stated here for divisibility, but can extract group structure, too.

Outline

Proportions

- $X \to S \to \mathbb{F}_{q_0}$ an abelian scheme of relative dimension g.

$$\mathcal{P}(X \to S, \ell, \mathbb{F}_q) = \frac{\left|\{s \in S(\mathbb{F}_q) : X_s[\ell](\mathbb{F}_q) \neq \{0\}\}\right|}{\left|S(\mathbb{F}_q)\right|}$$

$$\mathcal{Q}(X \to S, \ell, \mathbb{F}_q) = \frac{\left|\{s \in S(\mathbb{F}_q) : X_s[\ell](\mathbb{F}_q) \cong (\mathbb{Z}/\ell)^{2g}\}\right|}{\left|S(\mathbb{F}_q)\right|}.$$

Proportions

- $C \to S \to \mathbb{F}_{q_0}$ a relative smooth, proper curve.

$$\mathcal{P}(C \to S, \ell, \mathbb{F}_q) = \mathcal{P}(\mathrm{Jac}(C) \to S, \ell, \mathbb{F}_q)$$
$$\mathcal{Q}(C \to S, \ell, \mathbb{F}_q) = \mathcal{Q}(\mathrm{Jac}(C) \to S, \ell, \mathbb{F}_q)$$

Thus, $\mathcal{P}(C \to S, \ell, \mathbb{F}_q)$ is the proportion of elements with class number a multiple of ℓ.

Main result – connected monodromy

Theorem

Suppose $X \to S$ has connected ℓ-adic monodromy group, $q_0 \equiv 1 \bmod \ell$. If q sufficiently large, then:

$$\mathcal{Q}(X \to S, \ell, \mathbb{F}_q) > \frac{1}{\ell g(2g+1)}.$$

If $\ell \in \mathbb{L}$, a set of primes of positive density, then:

$$\mathcal{P}(X \to S, \ell, \mathbb{F}_q) > \frac{1}{\ell} - \mathcal{O}(1/\ell^2).$$

If monodromy group is known, get much more precise result.

Main result – unspecified monodromy

Theorem

Given $X \to S$, there exist β and ν such that if $q \equiv 1 \bmod \ell^\beta$ is sufficiently large, then:

$$\mathcal{Q}(X \to S, \ell, \mathbb{F}_q) > \frac{1}{\delta \ell^{g(2g+1)}}.$$

If $\ell \in \mathbb{L}$, a set of primes of density $\delta > 0$, then:

$$\mathcal{P}(X \to S, \ell, \mathbb{F}_q) > \frac{1}{\delta \ell} - \mathcal{O}(1/\ell^2).$$

There are bounds on β, ν and δ purely in terms of g.

Corollaries: ℓ-rank of class numbers often positive

In any reasonable family of function fields, the proportion of members for which the class group:

- has maximal ℓ-rank ($2g$) is positive;
- has an element of order ℓ is at least $1/\nu\ell$.

For universal families of curves, or of hyperelliptic curves, $\nu = 1$, $\delta = 1$, and the proportion members with an element of order ℓ in the class group is at least $1/\ell$.

Corollaries: Friedman-Washington conjecture

- Get a corrected, proven Friedman-Washington conjecture.
- Frequency with which H appears as ℓ-part of class group is inversely proportional to a "symplectic automorphism group" of H.

Recall: Friedman-Washington say this frequency should be $1/|\mathrm{Aut}(H)|$.

Corollaries: Cyclic covers of $\mathbb{P}^1$

- Let $\mathcal{H}_n$ be space of monic separable polynomials of degree n.
- If $f \in \mathcal{H}_n(\mathbb{F})$, let $K_{d,f} = \operatorname{Frac} \mathbb{F}[T, Y]/[Y^d - f(T)]$.
- $K_{d,f} = \mathbb{F}(C_{d,f})$, $C_{d,f}$ smooth, projective of genus $g = \frac{1}{2}((n-1)(d-1) + 1 - \gcd(d,n))$.
- If $q \gg_{\ell,d,n} 0$ then:

$$\frac{\left|\{f \in \mathcal{H}_n(\mathbb{F}_q) : \ell \,\big|\, \left|\mathrm{Cl}(K_{d,f})\right|\}\right|}{\left|\mathcal{H}_n(\mathbb{F}_q)\right|} > \frac{1}{2d\ell^{g(2g+1)}}.$$

This yields a lower bound for the density of such extensions in families.

ℓ-torsion: elliptic curves

- We know that
$$E[\ell](\overline{\mathbb{F}}) \cong (\mathbb{Z}/\ell)^2.$$
- $P \in E[\ell](\overline{\mathbb{F}})$ is actually defined over $\mathbb{F}$ if it is fixed by the Frobenius map Fr_E.
- Our question reduces to:

  Explain the distribution of Fr_E *in* $\mathrm{GL}_2(\mathbb{Z}/\ell)$ *as the choice of* E *varies.*

ℓ-torsion: curves of genus g

- We know that

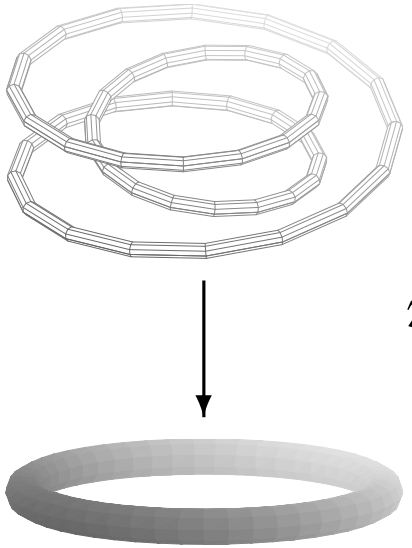
$$\mathrm{Jac}(C)[\ell](\overline{\mathbb{F}}) \cong (\mathbb{Z}/\ell)^{2g}.$$

- A $P \in \mathrm{Jac}(C)[\ell](\overline{\mathbb{F}})$ is actually defined over $\mathbb{F}$ if it is fixed by the Frobenius map Fr_C.
- Our question reduces to:

  Explain the distribution of Fr_C *in* $\mathrm{GL}_{2g}(\mathbb{Z}/\ell)$ *as the choice of* C *varies.*

Topological monodromy

A cover X of S corresponds to $\pi_1(S, s) \to \mathrm{Aut}(X_s)$.



$$\pi_1(S, s) = \langle \gamma \rangle \longrightarrow \mathrm{Aut}(T_s)$$

try it

Algebraic monodromy

- A variety S has a fundamental group, $\pi_1(S, s)$.
- A cover $X \to S$ corresponds to

$$\pi_1(S, s) \longrightarrow \mathrm{Aut}(X_s)$$

- A local system $\mathcal{F}$ of rank n Λ-modules corresponds to

$$\pi_1(S, s) \xrightarrow{\rho_{\mathcal{F}}} \mathrm{GL}(\mathcal{F}_s) \cong \mathrm{GL}_n(\Lambda)$$

- The monodromy group G of $\mathcal{F}$ is (the isomorphism class of the Zariski closure of) the image of this representation.

Frobenius

- $x \in S(k)$ gives

$$\pi_1(\operatorname{Spec} k) \xrightarrow{x_*} \pi_1(S, \eta)$$

- If $k = \mathbb{F}$ finite, set

$$\mathrm{Fr}_{x,\mathbb{F}} = x_*(a \mapsto a^q).$$

- Given $\mathcal{F}/X$, get $\rho_{\mathcal{F}}(\mathrm{Fr}_{x,\mathbb{F}}) \in \mathrm{GL}_n(\Lambda)$.

Equidistribution: finite

- Suppose monodromy group G finite.
- $W \subset G$ stable under conjugation.
- Theorem [Katz, Deligne]

$$\lim_{\#\mathbb{F}\to\infty} \frac{\#\{x \in S(\mathbb{F}) : \rho_{\mathcal{F}}(\mathrm{Fr}_{x,\mathbb{F}}) \in W\}}{\#S(\mathbb{F})} = \frac{\#W}{\#G}$$

- gap between left-hand and right-hand sides is $\sim \frac{1}{\sqrt{\#\mathbb{F}}}$.

Throughout, we'll assume $|\mathbb{F}| \equiv 1 \bmod \ell$.

Equidistribution: examples

- In triple cover of S^1

$$\pi_1(S^1, s) \xrightarrow{\rho} \mathrm{Aut}(T_s)$$

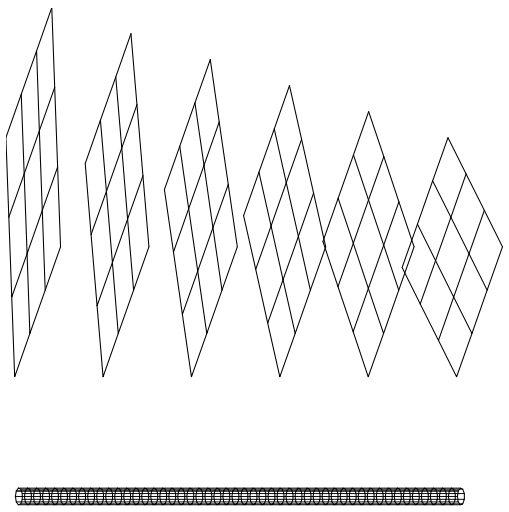
$$\mathbb{Z} \longrightarrow \mathrm{Sym}(1,2,3)$$

  image is $\{\mathrm{id}, (123), (132)\}$.
  For each g, $\rho^{-1}(g)$ has density $1/3$.
- Chebotarev: If L/K Galois, then local Frobenius elements are equidistributed in $\mathrm{Gal}(L/K)$.

Strategy

- Let $X \to S$ be a family of abelian varieties. Glue $X[\ell]$ together to get local system of $\mathbb{Z}/\ell$-vector spaces on S.

$$\begin{array}{c} \mathcal{E}[3] \\ \downarrow \\ S \end{array}$$



- $X_s[\ell] \neq 0 \iff \rho(\mathrm{Fr}_s - \mathrm{id})$ is not invertible.

Strategy

Study the mod $-\ell$ monodromy representation

$$\pi_1(S, \overline{\eta}_S) \xrightarrow{\rho} \mathrm{Aut}(X_{\overline{\eta}_S}[\ell]) \cong \mathrm{GL}_{2g}(\mathbb{F}_\ell) :$$

- Compute $G = \mathsf{M}(X \to S, \ell) = \rho(\pi_1(S, \overline{\eta}_S))$
- Calculate $W = \{g \in G : 1 \text{ is an eigenvalue of } g\}$.

Then

$$\frac{|\{s \in S(\mathbb{F}) : \ell \mid |X_s(\mathbb{F})|\}|}{|S(\mathbb{F})|} \approx \frac{|W|}{|G|}.$$

Monodromy group of curves: Examples

The monodromy group is know when the family is:

- $\mathcal{M}_g$, the universal family of curves of genus g; $G \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ [Deligne-Mumford].
- $\mathcal{H}_g$, the universal family of hyperelliptic curves of genus g; $G \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ [JK Yu, A.-Pries]
- $\mathcal{T}_g^\alpha$, a component of the universal family of cyclic cubic covers of $\mathbb{P}^1$ of genus g; $G^0 \cong \mathrm{U}^\alpha(\mathbb{Z}/\ell)$ for a certain unitary group U^α [A.-Pries]

Counting in $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$

Can write down a formula, by studying:

- structure of unipotent classes in $G(\overline{\mathbb{F}_\ell})$;
- how these classes behave over $\mathbb{F}_\ell$;
- the structure of their centralizers;
- and induction.

Sneak preview

g	r	$\alpha(g,r)$
3	0	$\frac{\ell^{12}-\ell^{11}-\ell^{10}+\ell^7+\ell^5+\ell^4-\ell^3-\ell-1}{(\ell^2-1)(\ell^4-1)(\ell^6-1)}$
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Summary

- If C is a hyperelliptic curve over a finite field $\mathbb{F}$, then the chance that $\ell | \# \operatorname{Jac}(C)[\ell](\mathbb{F})$ is about
$$\frac{1}{\ell - 1}.$$
- The proportion of hyperelliptic curves C for which $\operatorname{Jac}(C)[\ell](\mathbb{F}) \cong H$ is the proportion of $\gamma \in \operatorname{Sp}_{2g}(\mathbb{F}_\ell)$ for which
$$\ker(\gamma - \mathrm{id}) \cong H.$$

And that's the Friedman-Washington conjecture!

Outline

Analogy: E/K

Let E/K be an elliptic curve without complex multiplication. Serre proves:

- $T_\ell E = \lim_{\leftarrow n} E[\ell^n](K)$ has $\mathrm{Gal}(K)$-action.
- Characteristic polynomial of $\sigma \in \mathrm{Gal}(K)$ acting on $T_\ell E$ is in $\mathbb{Z}[X]$, and is independent of ℓ.
- For $\ell \gg 0$, image of $\mathrm{Gal}(K)$ in $\mathrm{Aut}(T_\ell(E))$ is large.

Monodromy group: Philosophy

- $\pi : X \to S$ proper smooth.
- Consider the sheaf $R^n \pi_* \mathbb{Q}_\ell$.
- Get a system of representations

$$\pi_1(S, \overline{\eta}) \xrightarrow{\rho_\ell} \mathrm{Aut}(H^i(X_{\overline{\eta}}, \mathbb{Q}_\ell))$$

which is *compatible*:
For $s \in S(\mathbb{F}_q)$, the characteristic polynomial of $\rho_\ell(\mathrm{Fr}_{s/\mathbb{F}_q})$ has $\mathbb{Z}$-coefficients, and is independent of ℓ.

and, Frobenius elements generate the fundamental group

Compatible systems

Conjecture Let $\{\mathcal{F}_\ell\}$ be a compatible system of representations of $\pi_1(S, s)$.

- There exists a number field E and a group G/F such that $\mathrm{M}(R^i\pi_*(\mathbb{Q}_\ell)) \otimes E_\lambda \cong G \otimes E_\lambda$.
- Moreover, the monodromy representation comes from a representation of G via base change.

Chin proves something very close to this for G^0.

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- Moreover, the monodromy representation comes from a representation of G via base change.

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Integral monodromy

- Serre: E an elliptic curve without CM, then image of Galois in $T_\ell(E)$ is $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for almost all ℓ.
- Conjecture Actual image of monodromy is hyperspecial in $G(E_\lambda)$, e.g., is $G(\mathcal{O}_\lambda)$, for almost all ℓ.
- Larsen proves this for a set of primes of density one.

Outline

Arbitrary families

- Use a general result of Larsen on compatible systems of Galois representations:
- Theorem [Larsen] For fixed $X \to S$, there exist a group G and a set of rational primes $\mathbb{L}$ of density one such that if $\ell \in \mathbb{L}$, then

$$\mathsf{M}(X \to S, \ell) \cong G(\mathbb{Z}/\ell).$$

Next step

Let $\mathbb{F} = \mathbb{F}_\ell$.

Theorem

$V/\mathbb{F}$ n-dimensional vector space, $G/\mathbb{F}$ connected split semisimple, $G \to \mathrm{GL}(V)$ a representation. If maximal $T_0 \subset G$ acts via a character which isn't a power of a root, then:

$$\frac{|\{\gamma \in G(\mathbb{F}) : \gamma \text{ fixes an element of } V\}|}{|G(\mathbb{F})|} > \frac{1}{\ell - 1} - \frac{\delta(n)}{\ell - 1}$$

where $\delta(n) = (n! + n)/(\ell - 1)$.

Sketch

- Fix maximally split torus $T_0 \subset G$.
- T_0 acts on V by character χ_0; at least $T_0(\mathbb{F})/(\ell - 1)$ elements fix something.
- Look for similar elements in other tori.

Example: GL_2

Two $\mathrm{GL}_2(\mathbb{F})$-conjugacy classes of tori:

- $T_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
- $T_1 = \begin{pmatrix} \alpha & \beta \\ \epsilon\beta & \alpha \end{pmatrix}$, for fixed ϵ with $\sqrt{\epsilon} \notin \mathbb{F}$.

Note that $T_1(\mathbb{F}) \cong \{\alpha + \beta\sqrt{\epsilon}\} \cong \mathbb{F}(\sqrt{\epsilon})^\times$.

T_1 is obtained from T_0 by twisting with $w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Example: GL_2

- Suppose $\chi_0 = \det : \mathrm{diag}(\lambda_1, \lambda_2) \mapsto \lambda_1\lambda_2$.
- Gives $\mathbb{F}$-rational character of T_1: $\gamma := \alpha + \beta\sqrt{\epsilon} \mapsto \mathsf{N}_{\mathbb{F}(\sqrt{\epsilon})/\mathbb{F}}(\gamma)$.
- Find $|T_i|/(\ell - 1)$ elements in each T_i.

Example: GL_2

- Suppose χ_0 is $\mathrm{diag}(\lambda_1, \lambda_2) \mapsto \lambda_1$.
- χ_0 *does not* twist to $\mathbb{F}$-rational character of T_1.
- Balanced by action of T_0 on V via $\chi_1 : \mathrm{diag}(\lambda_1, \lambda_2) \mapsto \lambda_2$.

So far...

- If ℓ-adic monodromy is hyperspecial in a split connected reductive group, then

$$\mathcal{P}(X \to S, \ell, q) > \frac{1}{\ell} - \mathcal{O}(1/\ell^2).$$

- Larsen: hyperspecial is density-one condition on primes ℓ.
- What if monodromy group isn't connected?

Disconnected monodromy

Lemma There exists étale Galois $\widetilde{S} \to S$ and extension $\mathbb{F}_{q_1}/\mathbb{F}_{q_0}$, so that image of

$$\pi_1(\widetilde{S}) \hookrightarrow \pi_1(S) \longrightarrow \mathrm{GL}(X_{\overline{\eta}}[\ell^\infty])$$

is connected.

- In fact, image is connected component of identity of original monodromy group.
- Existence of $\widetilde{S}$ for one ℓ is easy.
- Independence of ℓ due to Serre (and Larsen).
- Can find *a priori* bound on $\deg(S' \to S)$, $[\mathbb{F}_{q_1} : \mathbb{F}_{q_0}]$.

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Disconnected monodromy

Lemma

Suppose $X \to S$ has disconnected monodromy group $\mathsf{M}(X \to S, \ell) = G$, $\phi : \widetilde{S} \to S$ étale Galois of degree ν, $\mathsf{M}(X \times \widetilde{S} \to \widetilde{S}, \ell) = G^0$, $W \subset G^0$, then

$$\frac{|\{s \in S : \rho(\mathrm{Fr}_{X_s,\mathbb{F}}) \in W\}|}{|S(\mathbb{F}_q)|} > \frac{1}{\nu} \cdot \frac{|W|}{|G^0|}.$$

Ingredients:

- $\rho(\mathrm{Fr}_{X_s,\mathbb{F}})$ constant in fibers of ϕ.
- Equidistribution for $\widetilde{S} \to S$.

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Lemma

Suppose $X \to S$ has disconnected monodromy group $\mathsf{M}(X \to S, \ell) = G$, $\phi : \widetilde{S} \to S$ étale Galois of degree ν, $\mathsf{M}(X \times \widetilde{S} \to \widetilde{S}, \ell) = G^0$, $W \subset G^0$, then

$$\frac{|\{s \in S : \rho(\mathrm{Fr}_{X_s,\mathbb{F}}) \in W\}|}{|S(\mathbb{F}_q)|} > \frac{1}{\nu} \cdot \frac{|W|}{|G^0|}.$$

Ingredients:

- $\rho(\mathrm{Fr}_{X_s,\mathbb{F}})$ constant in fibers of ϕ.
- Equidistribution for $\widetilde{S} \to S$.

Outline

Cyclic covers of $\mathbb{P}^1$

Fix d, n, $\gcd(d, n) = 1$. Study

$$\mathcal{O}_{d,f} = \mathbb{F}_q[Y, T]/(Y^d - f(T))$$
$$h_{d,f} = \left|\mathrm{Cl}(\mathcal{O}_{d,f})\right|$$

where $f \in \mathcal{H}_n(\mathbb{F}_q)$, $\mathcal{H}_n$ being the space of monic, separable polynomials of degree n.
How is $h_{d,f}$ mod ℓ distributed?

Cyclic covers of $\mathbb{P}^1$

In special cases, we can also work out the error term:

Theorem

For ℓ in a set of positive density, if $q \equiv 1 \bmod \ell$, then

$$\frac{\left|\{f(T) \in \mathcal{H}_n(\mathbb{F}_q) : \ell | h(d,f)\}\right|}{\left|\mathcal{H}_n(\mathbb{F}_q)\right|} >$$

$$\frac{1}{2n}\left(\frac{1}{\ell} - \epsilon(g(d,n),\ell)\right) - \frac{2(2n)(n-1)!\left|\mathrm{Sp}_{2g(d,n)}(\mathbb{Z}/\ell)\right|}{\sqrt{q}}.$$

where $g(d,n) = \frac{1}{2}((n-1)(d-1)+1-\gcd(d,n))$.

Even without error term, strengthens work of Cardon, Murty, Chakraborty, Pacelli, Lee.

Cyclic covers of $\mathbb{P}^1$

- Let $C_{d,f}^{\mathrm{aff}} = \mathrm{Spec}(\mathcal{O}_{d,f})$.
- Then $C_{d,f}^{\mathrm{aff}}$ is open in $C_{d,f}$, a smooth, projective curve of genus g. It's a cyclic cover of the projective line.
- Hypothesis on d, n implies

$$\mathrm{Cl}(\mathcal{O}_{d,f}) \cong \mathrm{Cl}(\mathbb{F}_q(C_{d,f})) \cong \mathrm{Jac}(C_{d,f})(\mathbb{F}_q)$$

So, study the family $\mathrm{Jac}(C_d) \to \mathcal{H}_n$.
Let $G = \mathrm{M}(\mathrm{Jac}(\widetilde{C}_d) \to \mathcal{H}_n, \ell)$.

Cyclic covers of $\mathbb{P}^1$

- Error in equidistribution is of the form $2|G|B/\sqrt{q}$, where B is such that if $\phi : Y \to \mathcal{H}_n$ Galois, étale, $p \nmid \deg \phi$, then

$$\sigma_c(Y) := \sum_i \dim H_c^i(Y, \overline{\mathbb{Q}}_\ell) \leq \deg \phi \cdot B.$$

- Estimate B by pulling back to $\widetilde{\mathcal{H}}_n$:

$$\begin{tikzcd} \widetilde{\mathcal{H}}_n = \mathbb{A}^n - \{z_i = z_j : i \neq j\} \arrow[d] & (a_1, \cdots, a_n) \arrow[d, mapsto] \\ \mathcal{H}_n & \prod (T - a_n) \end{tikzcd}$$

Cyclic covers of $\mathbb{P}^1$

For sums of Betti numbers of covers of $\widetilde{\mathcal{H}}_n$, use general result on hyperplane arrangements:

Lemma

Let $\mathcal{A}$ be a hyperplane arrangement in a vector space V with complement $\mathcal{M}(\mathcal{A})$. Let $\phi : Y \to \mathcal{M}(\mathcal{A})$ be an irreducible étale tame Galois cover. Then

$$\sigma_c(Y) \leq (\deg \phi)\sigma_c(\mathcal{A}).$$

Ingredients:

- Poincaré duality: $\sigma_c(Y) = \sigma(Y)$.
- Deligne and Illusie: If $\phi : Y \to X$ étale, then $\chi(Y) = (\deg \phi)\chi(X)$, where $\chi(U) = \sum_i (-1)^i h^i(U)$.
- Induction on $\dim V$.

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Hyperplane complements

- Let V be an n-dimensional vector space.
- Let $\mathcal{A} = \{X_1, \cdots, X_r\}$ be a finite set of hyperplanes in V.
- Goal: understand cohomology ring of $\mathcal{M}(\mathcal{A}) = V - \cup_{X \in \mathcal{A}} X$.

Hyperplane complements

- Construct $\mathcal{L}(A)$, lattice of intersections of elements of $\mathcal{A}$, ordered by inclusion.
- Let μ be the Möbius function of $\mathcal{L}(A)$.
- The rank of an element $X \in \mathcal{L}(A)$ is $r_{\mathcal{A}}(X) = \operatorname{codim}_V(X)$.

- Cohomology groups of $\mathcal{M}(\mathcal{A})$ given by

$$\dim H^i(\mathcal{M}(\mathcal{A}), \overline{\mathbb{Q}}_\ell) = (-1)^i \sum_{X \in \mathcal{L}(\mathcal{A}) : r_{\mathcal{A}}(X) = i} \mu(X).$$

In particular, note that i^{th} Betti number depends only on elements of $\mathcal{L}(\mathcal{A})$ with codimension at most i in V.

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Hyperplane complements

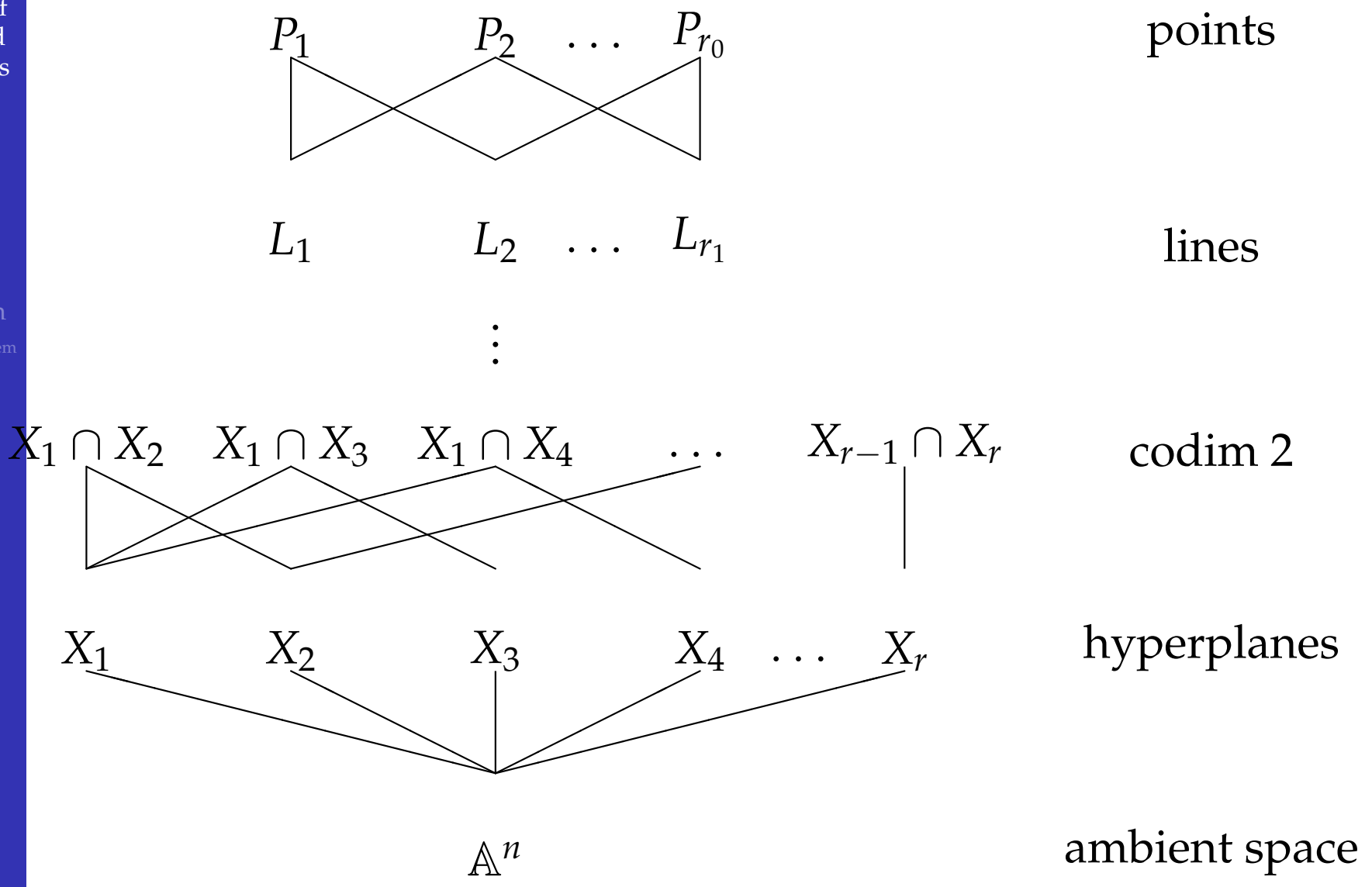
- Let $H \subset V$ be a hyperplane.
- Define $\mathcal{A}_H = \{H \cap X : X \in \mathcal{A}\}$, an arrangement in H.
- If H is *generic*, $X \in \mathcal{L}(\mathcal{A})$ of positive dimension, then

$$r_{\mathcal{A}}(X) = r_{\mathcal{A}_H}(X_H);$$

  $\mathcal{L}(\mathcal{A}_H)$ obtained from $\mathcal{L}(\mathcal{A})$ by removing top row.
- Conclusion: If $H \subset V$ generic, then

$$h^i(\mathcal{M}(\mathcal{A}_H), \overline{\mathbb{Q}}_\ell) = \begin{cases} h^i(\mathcal{M}(\mathcal{A}), \overline{\mathbb{Q}}_\ell) & 0 \le i \le n-1 \\ 0 & i = n. \end{cases}.$$

$\mathcal{L}(\mathcal{A})$



$\mathcal{L}(\mathcal{A}_H)$

$L_1 \cap H \quad L_2 \cap H \ldots L_{r_1} \cap H$ points

$\vdots$

$X_{12} \cap H \quad X_{13} \cap H \quad X_{14} \cap H \quad \ldots \quad X_{r-1} \cap X_r$ codim 2

$X_1 \cap H \quad X_2 \cap H \quad X_3 \cap H \quad X_4 \cap H \ldots X_r \cap H$ hyperplanes

H ambient space

Conclusion

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