PROBLEM SESSION ON LOCAL-GLOBAL PRINCIPLES

1. Let $T = k[[t]]$ and let $\mathcal{X} = \mathbb{P}^1_T$, with closed fiber X. Let $P \in X$ be the point $x = \infty$ and let U be the complement of P. Let $\wp$ be the branch of $\mathcal{X}$ at P. Let F be the function field of $\mathcal{X}$.

a) Prove that $F = F_P \cap F_U$ in $F_\wp$.

b) Explain why this says that the inverse limit of the inverse system consisting of $F_P, F_U, F_\wp$ is F.

2. What does the matrix factorization result say about elements of $\mathrm{GL}_n(F_\wp)$ in the situation of problem 1 above? In particular, in the case of $n = 1$ and the 1×1 matrix whose entry is $1 + xt + x^{-1}t$, what matrices does it say must exist? Find such matrices explicitly. (For the last part, you may stop modulo t^r for some r once the situation is clear. But explain what is meant by "modulo t^r", given that we are working over fields.)

3. In the situation of problem 1 above, consider the discrete valuations v on F respectively corresponding to $(t), (x), (x - t), (x - 1)$. Show that each of the associated completions F_v must contain either F_P or F_U.

4. Let T be a complete discrete valuation ring and let $\mathcal{X}$ be a regular projective T-curve whose closed fiber X has the property that there are no more than two branches at any point (e.g. if each irreducible component of X is smooth and no more than two components meet at any point). Define the *Deligne-Mumford reduction graph* as follows: For each irreducible component C of X, take a vertex v_C. And for each point P of X at which there is more than one branch, take an edge e_P, whose vertices correspond to the components that pass through P.

a) Show that the barycentric subdivision of this graph is the bipartite reduction graph that we have discussed, and is therefore homotopy equivalent to that graph as a topological space, having the same fundamental group.

b) Explain why the Deligne-Mumford reduction graph requires the above hypotheses in order to be defined.

5. Find explicit equations for a regular projective curve $\mathcal{X}$ over a complete discrete valuation ring T such that the closed fiber has more than one irreducible component and is therefore singular. (Here *regular* means that each local ring is regular. An easier version of the problem: replace "regular" by "normal", meaning that each local ring is an integrally closed domain.)

6. Find a complete discrete valuation ring T, and a regular (or at least normal) projective curve $\mathcal{X}$ over T with function field F, together with a binary quadratic form q over F such that q is isotropic over F_v for every discrete valuation v on F but q is not isotropic over F.

7. Let $T = k[[t]]$ (for some field k) and define $\mathcal{X} \to \mathbb{P}^1_T$ on open patches via

$$\begin{aligned} y^2 &= x(x-1)(1-xt) \quad &&\text{over } k[[t]][x] \\ z^2 &= x^{-1}(1-x^{-1})(x^{-1}-t) \quad &&\text{over } k[[t]][x^{-1}] \\ y &= zx^2 \quad &&\text{over } k[[t]][x, x^{-1}] \end{aligned}$$

Check that this is well defined, find equations for the general fiber and the closed fiber X and draw schematic pictures. Try to draw $\mathcal{X}$.

8. Show that the morphism defined in problem 7 maps the only "bad" point P' of the closed fiber X to the point P in 1. Compute $F_{P'}$ and the corresponding $F_{U'}$ for all components U' of $X \setminus \{P'\}$. On $\mathcal{X}$, there are two branches $F_{\wp_1}$ and $F_{\wp_2}$. Compute those fields and draw the reduction graph of $\mathcal{X}$. Show that in each of those branch fields, $1 - xt$ has a square root. Now simultaneously factor the element $(\alpha, 1) \in \mathrm{GL}_1(F_{\wp_1}), \mathrm{GL}_1(F_{\wp_2})$ where α is a square root of $1 - xt$.

9. (Suggested by Diego Izquierdo)
Let X/k be a curve such that $X_{\bar{k}} \cong \mathbb{P}^1_{\bar{k}}$. Let $K = k(X)$. Define

$$Ш^2(K, \mathbb{G}_m) := \mathrm{Ker}\left(\mathrm{Br}(K) \to \prod_{v \in X^{(1)}} \mathrm{Br}(K_v)\right),$$

where $X^{(1)}$ denotes the set of closed points of X. Assuming that $Ш^2(K, \mathbb{G}_m)$ is divisible, show that it must vanish.

The last two problems are open problems suggested by Asher Auel.

10. Let X be a smooth projective surface over an algebraically closed field k of characteristic $\neq 2$ and let K be its function field. Is it true that $Ш^1(K, \mathbb{Z}/2\mathbb{Z}) = 0$? Equivalently, if $Y \to X$ is a nonsplit étale cover of degree 2, does there exist a codimension 1 point of X that does not split in Y?

11. Let X be a curve over a totally imaginary number field k and let K be its function field. Construct a locally isotropic, yet globally anisotropic, quadratic form in 4 variables over K. (To do this, letting $\mathcal{X}$ be a regular model of X over the ring of integers of k, it is sufficient to prove the existence of a branched double cover $\mathcal{Y} \to \mathcal{X}$, with regular branch divisor, such that $\mathrm{Br}(\mathcal{Y})$ contains a nontrivial class represented by a quaternion algebra.)