T/F 1. If \( w \neq 0, v \neq 0, \) and \( w \times v = z, \) then \( v \times w = z. \) ♦

False. Counterexample: If \( w = i \neq 0 \) and \( v = j \neq 0, \) then \( w \times v = k \) but \( v \times w = -k. \) Note that, since \( w \times v = -v \times w, \) any non-parallel \( w, \) \( v \) constitute a counterexample. ♦

T/F 2. If a non-zero vector \( v \) lies in a plane \( P, \) and \( w \) is a non-zero vector not in \( P, \) then \( w \) is orthogonal to \( P \) only if \( v \cdot w = 0. \) ♦

True. Proof: We must show that \( w \perp P \) implies that \( v \cdot w = 0. \)

\( w \perp P \) means that \( w \) is orthogonal to every vector in \( P. \) Since \( v \in P, \) \( w \) is then orthogonal to \( v. \) Therefore, \( v \cdot w = 0. \) ♦

T/F 3. The cross product of any two unit vectors is also a unit vector. ♦

False. Counterexample: \( i \times i = 0. \) Note that any non-orthogonal unit vectors also constitute a counterexample. ♦

T/F 4. If \( A, B, C \) are non-zero vectors in \( \mathbb{R}^3 \) and \( A \times (B \times C) = 0, \) then \( A, B, \) and \( C \) are parallel. ♦

False. Counterexample: If \( A = i \neq 0, B = j \neq 0, \) and \( C = k \neq 0, \) then \( A \times (B \times C) = i \times (j \times k) = i \times i = 0. \) Note that if \( A \times (B \times C) = 0 \) then \( A \) is in fact perpendicular to both \( B \) and \( C. \) ♦

T/F 5. For any vectors \( u \) and \( v \) in \( \mathbb{R}^3, \) \( (u \times v) \cdot u > 0. \) ♦

False. Since \( u \times v \) is perpendicular to \( u, \) \( (u \times v) \cdot u \) is always equal to 0. ♦

MC 1. The parametric equations of a curve are \( x = t^2, y = t^4, t \geq 0; \) the Cartesian equation is: a) \( y = x^2; \) b) \( y = \sqrt{x}; \) c) \( y = 2x^2; \) d) \( y = \sqrt{2x}; \) e) \( y = x^2/2; \) f) \( y = \sqrt{x/2}. \) ♦

Since \( y = t^4 = (t^2)^2 = x^2, \) the correct answer is a) \( y = x^2. \) ♦

MC 2. What is the slope of the tangent to the curve \( x = \sin t, y = \cos t \) when \( t = \pi/3? \)

a) \( 1/\sqrt{2}; \) b) \( 1/\sqrt{3}; \) c) \( -1/\sqrt{2}; \) d) \( -1/\sqrt{3}; \) e) \( \sqrt{2}; \) f) \( -\sqrt{3}. \) ♦

Calculus II
Since
\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t},
\]
the slope of (the tangent to) the curve at \( t = \pi/3 \) is
\[
\frac{-\sin(\pi/3)}{\cos(\pi/3)} = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}.
\]

Hence, the correct answer is f) \(-\sqrt{3}\). ♦

**MC 3.** Find the length of the curve whose parametric equations are \( x = \ln \cos t, y = t, 0 \leq t \leq \pi/4 \). a) \( \ln(2\sqrt{2} + 1) \); b) \( \ln(\sqrt{2} + 1) \); c) \( \ln(2\sqrt{2} + 2) \); d) \( \ln(\sqrt{2} + 1) \); e) \( \ln(2\sqrt{3} + 1) \); f) \( \ln(\sqrt{3} + 2) \). ♦

The length of a parametrized curve is
\[
L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt.
\]

Applying this formula with \( dx/dt = -\sin t / \cos t = -\tan t \) and \( dy/dt = 1 \) gives
\[
L = \int_0^{\pi/4} \sqrt{\tan^2 t + 1} \, dt
= \int_0^{\pi/4} \sqrt{\sec^2 t} \, dt
= \int_0^{\pi/4} \sec t \, dt.
\]

(The last step is valid since \( \sec t \geq 0 \) between \( t = 0 \) and \( \pi/4 \).) From here there are two methods to evaluate this integral.

Method 1: This is my method, and involves dogged substitutions. The sub-
substitutions I use are \( u = \cos t \), \( v = 1 - u^2 \), and \( w = \sqrt{v} \).

\[
L = \int_{0}^{\pi/4} \sec t \, dt
\]

\[
= - \int_{1}^{1/\sqrt{2}} \frac{du}{u \sqrt{1 - u^2}}
\]

\[
= \frac{1}{2} \int_{0}^{1/\sqrt{2}} \frac{dv}{(1 - v) \sqrt{v}}
\]

\[
= \frac{1}{2} \int_{0}^{1/\sqrt{2}} \frac{2}{1 - w^2} \, dw
\]

\[
= \frac{1}{2} \left( - \ln|1 - w| + \ln|1 + w| \right)_{0}^{1/\sqrt{2}}
\]

\[
= \frac{1}{2} \left( - \ln \left( \frac{1}{\sqrt{2}} \right) + \ln \left( 1 + \frac{1}{\sqrt{2}} \right) \right)
\]

\[
= \frac{1}{2} \left( \frac{\ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)}{\sqrt{2}} \right)
\]

\[
= \frac{1}{2} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)
\]

\[
= \ln \sqrt{3 + 2 \sqrt{2}}.
\]

Now, \((1 + \sqrt{2})^2 = 3 + 2 \sqrt{2}\), so

\[
L = \ln \sqrt{3 + 2 \sqrt{2}}
\]

\[
= \ln(1 + \sqrt{2}).
\]

Hence, the correct answer is d) \( \ln(\sqrt{2} + 1) \). ♦

Method 1': This method rolls the three substitutions of Method 1 into one less obvious substitution \( u = \sin t \).

\[
L = \int_{0}^{\pi/4} \sec t \, dt
\]

\[
= \int_{0}^{\pi/4} \frac{\cos t}{1 - \sin^2 t} \, dt
\]

\[
= \int_{0}^{1/\sqrt{2}} \frac{du}{1 - u^2}.
\]

From this point forward, Method 1' is the same as Method 1. ♦
Method 2: I don’t think anyone short of a true genius would know this method without seeing it at least once before. It uses the substitution \( u = \sec t + \tan t \).

\[
L = \int_0^{\pi/4} \sec t \, dt \\
= \int_0^{\pi/4} \frac{\sec t(\sec t + \tan t)}{\sec t + \tan t} \, dt \\
= \int_0^{\pi/4} \frac{\sec t\tan t + \sec^2 t}{\sec t + \tan t} \, dt \\
= \int_1^{1+\sqrt{2}} \frac{du}{u} \\
= (\ln|u|)\bigg|_1^{1+\sqrt{2}} \\
= \ln|1 + \sqrt{2}| = \ln(1 + \sqrt{2}).
\]

Hence, the correct answer is d) \( \ln(\sqrt{2} + 1) \).

**MC 4.** Find the Cartesian equation for the curve given by the polar equation \( r \sin \theta + r^2 \cos^2 \theta + r^2 = 0 \).

a) \( 2x + 2x^2 + y^2 = 0 \); b) \( x^2 + y + y^2 = 0 \); c) \( 2x^2 + y + y^2 = 0 \); d) \( x^2 + 2y + 2y^2 = 0 \); e) \( x^2 + x + y^2 = 0 \); f) \( x + x^2 + 2y^2 = 0 \).

Substituting \( r \sin \theta = y \), \( r \cos \theta = x \), and \( r^2 = x^2 + y^2 \), we get

\[
y + x^2 + (x^2 + y^2) = 0 \\
2x^2 + y + y^2 = 0.
\]

Hence, the correct answer is c) \( 2x^2 + y + y^2 = 0 \).

**MC 5.** Find the area of the region inside the polar curve \( r = \sin \theta \).

Method 1: The picture on the exam sheet suggests that this curve is a circle of radius 1/2 centered at \((x, y) = (0, 1/2)\). Indeed, we can prove this by converting to Cartesian coordinates:

\[
r = \sin \theta \\
r^2 = r \sin \theta \\
\]

\[
x^2 + y^2 = y \\
x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \\
x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.
\]

The area of a circle of radius \( r \) is \( \pi r^2 \), so the area of this circle is \( \pi (1/2)^2 = \pi / 4 \). Hence, the correct answer is e) \( \pi / 4 \).
Method 2: By inspection, one loop of this curve is parametrized by \( r = \sin \theta, 0 \leq \theta \leq \pi \). Then by the polar formula for area, the area within this curve is

\[
\int_0^\pi \frac{r^2}{2} d\theta = \int_0^\pi \frac{\sin^2 \theta}{2} d\theta \\
= \int_0^\pi \frac{1}{2} \frac{1 - \cos 2\theta}{2} d\theta \\
= \int_0^\pi \frac{1}{4} d\theta - \frac{1}{4} \int_0^\pi \cos 2\theta d\theta \\
= \frac{\pi}{4} - \frac{1}{8} (\sin 2\theta) \bigg|_0^\pi \\
= \frac{\pi}{4}.
\]

(If you haven’t memorized the identity \( \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \), you can get it by solving the simultaneous equations \( \sin^2 \theta + \cos^2 \theta = 1, \cos^2 \theta - \sin^2 \theta = \cos 2\theta \) for \( \sin^2 \theta \).) Hence, the correct answer is e) \( \pi/4 \). ♦

MC 6. Find the area of the shaded region inside both of the polar curves \( r = 4 \sin \theta \) and \( r = 4 \cos \theta \). ♦

Method 1: By the argument of Method 1 of problem 5, the two curves are circles of radius 2, one centered at the point \( P_1 = (x, y) = (2, 0) \) and \( P_2 = (x, y) = (0, 2) \). The circles (call the one centered on the x-axis \( C_1 \) and the other \( C_2 \)) meet at the origin \( O \) and at the point \( Q = (x, y) = (2, 2) \). The given region is the (disjoint union) of the segment \( OQ \) of \( C_1 \) and the segment \( OQ \) of \( C_2 \).

The area of the segment \( OQ \) of \( C_1 \) is the difference between the area of the sector \( P_1OQ \) and the area of the triangle \( P_1OQ \). Since the angle between \( P_1O \) and \( P_1Q \) is \( \pi/2 \), the area of the sector \( P_1OQ \) is

\[
\frac{\pi/2}{2\pi} \pi(2)^2 = \pi.
\]

By the well-known formula for the area of a triangle, the area of the triangle \( P_1OQ \) is \( 2(2)/2 = 2 \). Hence, the area of the segment \( OQ \) of \( C_1 \) is \( \pi - 2 \). Similarly, the area of the segment \( OQ \) of \( C_2 \) is \( \pi - 2 \). Hence, the area of the shaded region is \( 2(\pi - 2) = 2\pi - 4 \), so the correct answer is a) or f) \( 2\pi - 4 \). ♦

Method 2: The part of the region below the line \( y = x \) (in polar coordinates,
\( \theta = \pi/4 \) is defined by \( 0 \leq r \leq 4 \sin \theta \) and \( 0 \leq \theta \leq \pi/4 \). Its area is then
\[
\int_0^{\pi/4} \left( \frac{4 \sin \theta}{2} \right)^2 \, d\theta = 8 \int_0^{\pi/4} \sin^2 \theta \, d\theta \\
= 4 \int_0^{\pi/4} 1 - \cos 2\theta \, d\theta \\
= 4(\theta - (\sin 2\theta)/2)|_0^{\pi/4} \\
= 4 \left( \frac{\pi}{4} - \frac{1}{2} \right) \\
= \pi - 2.
\]

Likewise, the part of the region above the line \( y = x \) is defined by \( 0 \leq r \leq 4 \cos \theta \) and \( \pi/4 \leq \theta \leq \pi/2 \). Its area is then
\[
\int_{\pi/4}^{\pi/2} \left( \frac{4 \cos \theta}{2} \right)^2 \, d\theta = 8 \int_{\pi/4}^{\pi/2} \cos^2 \theta \, d\theta \\
= 4 \int_{\pi/4}^{\pi/2} 1 + \cos 2\theta \, d\theta \\
= 4(\theta + (\sin 2\theta)/2)|_{\pi/4}^{\pi/2} \\
= 4 \left( \frac{\pi}{2} + 0 - \frac{\pi}{4} - \frac{1}{2} \right) \\
= \pi - 2.
\]

Hence, the area of the entire region is \( 2\pi - 4 \), and the correct answer is a) or f) \( 2\pi - 4 \). ♦

**MC 7.** Find a unit vector in the direction of \( \mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \). a) \( \mathbf{i} + \mathbf{k} \); b) \( \mathbf{i} \); c) \( \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \); d) \( \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k} \); e) \( \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k} \); f) \( \mathbf{i} + \mathbf{j} \). ♦

Since
\[
|\mathbf{v}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3,
\]
the vector \( \mathbf{v}/3 = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \) is a unit vector in the direction of \( \mathbf{v} \). Hence, the correct answer is c) \( \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \). ♦

**MC 8.** For what value of \( x \) will the vectors \( \mathbf{v} = \langle x, 3 \rangle \) and \( \mathbf{w} = \langle 4, 5 \rangle \) be orthogonal? a) \( -4/15 \); b) \( 15/4 \); c) \( 20/3 \); d) \( -15/4 \); e) \( 3/20 \); f) \( 4/15 \). ♦

\( \mathbf{v} \perp \mathbf{w} \) when \( \mathbf{v} \cdot \mathbf{w} = 0 \). Since \( \mathbf{v} \cdot \mathbf{w} = x \cdot 4 + 3 \cdot 5 = 4x + 15 \), we solve \( 4x + 15 = 0 \): \( x = -15/4 \). Hence, the correct answer is d) \( -15/4 \). ♦

**MC 9.** Find the length of the cardioid \( r = 1 + \sin \theta \). a) \( 2\pi \); b) \( \pi/2 \); c) \( \pi \); d) 0; e) 8; f) \( 3\pi/2 \); g) 16. ♦
The length of a curve given in polar coordinates is

\[ L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta. \]

Applying this formula to the cardioid \( r = 1 + \sin \theta \) from \( \theta = 0 \) to \( 2\pi \) gives

\[
L = \int_0^{2\pi} \sqrt{(1 + \sin \theta)^2 + \cos^2 \theta} \, d\theta \\
= \int_0^{2\pi} \sqrt{1 + 2 \sin \theta + \sin^2 \theta + \cos^2 \theta} \, d\theta \\
= \int_0^{2\pi} \sqrt{2 + 2 \sin \theta} \, d\theta.
\]

From here, I see two ways to evaluate this integral.

Method 1:

\[
L = \int_0^{2\pi} \sqrt{2 + 2 \sin \theta} \, d\theta \\
= \int_0^{2\pi} \sqrt{2 + 2 \cos(\theta - \pi/2)} \, d\theta \\
= \int_0^{2\pi} 2 \sqrt{\frac{1 + \cos(\theta - \pi/2)}{2}} \, d\theta \\
= \int_0^{2\pi} 2 \sqrt{\cos^2(\theta/2 - \pi/4)} \, d\theta \\
= \int_0^{3\pi/2} 2 \cos(\theta/2 - \pi/4) \, d\theta - \int_{3\pi/2}^{2\pi} 2 \cos(\theta/2 - \pi/4) \, d\theta \\
= 4 \sin(\theta/2 - \pi/4) \bigg|_0^{3\pi/2} - 4 \sin(\theta/2 - \pi/4) \bigg|_{3\pi/2}^{2\pi} \\
= 4(1 - (-2\sqrt{2})) - 4(2 \sqrt{2} - 1) \\
= 8.
\]

Hence, the correct answer is e) 8. ♦

Feb. 7\textsuperscript{th} 2005
Method 2: Using the substitution \( u = 1 - \sin \theta \) with \( du = -\cos \theta \, d\theta \),

\[
L = \int_0^{2\pi} \sqrt{2 + 2 \sin \theta} \, d\theta \\
= \sqrt{2} \int_0^{2\pi} \sqrt{\frac{1 - \sin^2 \theta}{1 - \sin \theta}} \, d\theta \\
= \sqrt{2} \int_0^{2\pi} \sqrt{\frac{\cos^2 \theta}{1 - \sin \theta}} \, d\theta \\
= \sqrt{2} \int_{-\pi/2}^{3\pi/2} \sqrt{\frac{\cos^2 \theta}{1 - \sin \theta}} \, d\theta \\
= \sqrt{2} \int_{-\pi/2}^{\pi/2} \sqrt{\frac{\cos \theta}{1 - \sin \theta}} \, d\theta + \sqrt{2} \int_{\pi/2}^{3\pi/2} \sqrt{\frac{\cos \theta}{1 - \sin \theta}} \, d\theta \\
= -\sqrt{2} \int_0^2 \frac{du}{\sqrt{u}} + \sqrt{2} \int_0^2 \frac{du}{\sqrt{u}} \\
= 4\sqrt{2} \int_0^2 \frac{du}{2\sqrt{u}} \\
= 4\sqrt{2}(\sqrt{u})_0^2 \\
= 4\sqrt{2} \cdot 2 = 8.
\]

Hence, the correct answer is e) 8.

**MC 10.** Find a unit vector orthogonal to both \((1, -1, 0)\) and \((1, 2, 3)\). a) \(\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle\); b) \(\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle\); c) \(\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle\); d) \(\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle\); e) \(\langle \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle\); f) \(\langle \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle\).

Let \(a = (1, -1, 0)\) and \(b = (1, 2, 3)\). Then the cross product \(a \times b\) is orthogonal to both \(a\) and \(b\).

\[
a \times b = (1 \cdot 2 - 0 \cdot 1)i - (1 \cdot 3 - 0 \cdot 1)j + (1 \cdot 2 - (-1) \cdot 1)k = -3i - 3j + 3k.
\]

The magnitude of this cross product is \(\sqrt{(-3)^2 + (-3)^2 + 3^2} = 3\sqrt{3}\), so the unit vectors

\[
\pm \frac{1}{3\sqrt{3}} (-3, -3, 3) = \pm \left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle
\]

are orthogonal to both \(a\) and \(b\). Hence, the correct answer is d) \(\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle\).

**Free 1.** Find the area of the surfaces formed by rotating the given curves about the specified axis.

a. [Redacted.]

b. \(r = \sqrt{\cos 2\theta}, 0 \leq \theta \leq \pi/4\) about the directrix (x-axis).
The surface area of a surface of revolution about the x-axis given in polar coordinates is

\[ SA = \int_a^b 2\pi r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta. \]

Since \( \frac{dr}{d\theta} = -\sin 2\theta / \sqrt{\cos 2\theta} \), applying this formula gives

\[ SA = \int_0^{\pi/4} 2\pi \cos 2\theta \sin \theta \sqrt{\cos 2\theta + \sin^2 2\theta / \cos 2\theta} d\theta \]

\[ = \int_0^{\pi/4} 2\pi \sin \theta \cos^2 2\theta + \sin^2 2\theta d\theta \]

\[ = \int_0^{\pi/4} 2\pi \sin \theta d\theta \]

\[ = 2\pi (-\cos \theta) \bigg|_0^{\pi/4} \]

\[ = 2\pi \left( -\frac{\sqrt{2}}{2} + 1 \right) \]

\[ = \pi (2 - \sqrt{2}). \]

2. Below are five polar/parametric graphs and a selection of equations. Match each graph with the equation that produces it.

A. \((3) r \cos \theta = 1.\)
B. \((1) r = \cos 2\theta.\)
C. \((4) r = 1 + \cos \theta.\)
D. \((2) r = \theta.\)
E. \((5) r = -\sin \theta.\)

3.a. Find a vector orthogonal to the plane containing the points \( P = (1,0,-1), Q = (2,4,5), \) and \( R = (3,1,7). \)

We need a vector orthogonal to the vectors \( \overrightarrow{PQ} = (1,4,6) \) and \( \overrightarrow{PR} = (2,1,8) \) which are tangent to the plane \( PQR. \) To find such a vector, we take the cross product

\[ \overrightarrow{PQ} \times \overrightarrow{PR} = (4\cdot8 - 6\cdot1)i - (1\cdot8 - 2\cdot6)j + (1\cdot1 - 4\cdot2)k = 26i + 4j - 7k. \]

b. What is the area of the triangle whose vertices are \( P, Q, \) and \( R? \)
The area is half the magnitude of the cross product:

\[
A = \frac{|\overrightarrow{PQ} \times \overrightarrow{PR}|}{2}
\]

\[
= \frac{\sqrt{26^2 + 4^2 + (-7)^2}}{2}
\]

\[
= \frac{\sqrt{676 + 16 + 49}}{2}
\]

\[
= \frac{\sqrt{741}}{2} \quad \diamondsuit
\]

4. Show that the vector \( \mathbf{v} = \mathbf{b} - \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} \) is orthogonal to \( \mathbf{a} \). \( \diamondsuit \\

We must show that \( \mathbf{v} \cdot \mathbf{a} = 0 \).

\[
\mathbf{v} \cdot \mathbf{a} = \left( \mathbf{b} - \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} \right) \cdot \mathbf{a}
\]

\[
= \mathbf{b} \cdot \mathbf{a} - \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} \cdot \mathbf{a}
\]

\[
= \mathbf{a} \cdot \mathbf{b} - \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) |\mathbf{a}|^2
\]

\[
= \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}
\]

\[
= 0. \quad \blacklozenge
\]