Midterm 1

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T/F 1. If $\mathbf{w} \neq \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$, and $\mathbf{w} \times \mathbf{v} = \mathbf{z}$, then $\mathbf{v} \times \mathbf{w} = \mathbf{z}$. ♢

False. Counterexample: If $\mathbf{w} = \mathbf{i} \neq \mathbf{0}$ and $\mathbf{v} = \mathbf{j} \neq \mathbf{0}$, then $\mathbf{w} \times \mathbf{v} = \mathbf{k}$ but $\mathbf{v} \times \mathbf{w} = -\mathbf{k}$. Note that, since $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$, any non-parallel $\mathbf{w}, \mathbf{v}$ constitute a counterexample. ♦

T/F 2. If a non-zero vector $\mathbf{v}$ lies in a plane P, and $\mathbf{w}$ is a non-zero vector not in P, then $\mathbf{w}$ is orthogonal to P only if $\mathbf{v} \cdot \mathbf{w} = 0$. ♢

True. Proof: We must show that $\mathbf{w} \perp P$ implies that $\mathbf{v} \cdot \mathbf{w} = 0$. $\mathbf{w} \perp P$ means that $\mathbf{w}$ is orthogonal to every vector in P. Since $\mathbf{v} \in P$, $\mathbf{w}$ is then orthogonal to $\mathbf{v}$. Therefore, $\mathbf{v} \cdot \mathbf{w} = 0$. ♦

T/F 3. The cross product of any two unit vectors is also a unit vector. ♢

False. Counterexample: $\mathbf{i} \times \mathbf{i} = \mathbf{0}$. Note that any non-orthogonal unit vectors also constitute a counterexample. ♦

T/F 4. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are non-zero vectors in $\mathbb{R}^3$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{0}$, then $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are parallel. ♢

False. Counterexample: If $\mathbf{A} = \mathbf{i} \neq \mathbf{0}$, $\mathbf{B} = \mathbf{j} \neq \mathbf{0}$, and $\mathbf{C} = \mathbf{k} \neq \mathbf{0}$, then $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = \mathbf{i} \times \mathbf{i} = \mathbf{0}$. Note that if $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{0}$ then $\mathbf{A}$ is in fact perpendicular to both $\mathbf{B}$ and $\mathbf{C}$. ♦

T/F 5. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} > 0$. ♢

False. Since $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$ is always *equal* to 0. ♦

MC 1. The parametric equations of a curve are $x = t^2, y = t^4, t \geq 0$; the Cartesian equation is: a) $y = x^2$; b) $y = \sqrt{x}$; c) $y = 2x^2$; d) $y = \sqrt{2x}$; e) $y = x^2/2$; f) $y = \sqrt{x/2}$. ♢

Since $y = t^4 = (t^2)^2 = x^2$, the correct answer is a) $y = x^2$. ♦

MC 2. What is the slope of the tangent to the curve $x = \sin t, y = \cos t$ when $t = \pi/3$? a) $1/\sqrt{2}$; b) $1/\sqrt{3}$; c) $-1/\sqrt{2}$; d) $-1/\sqrt{3}$; e) $\sqrt{2}$; f) $-\sqrt{3}$. ♢

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t},$$

the slope of (the tangent to) the curve at $t = \pi/3$ is

$$\frac{-\sin(\pi/3)}{\cos(\pi/3)} = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}.$$

Hence, the correct answer is f) $-\sqrt{3}$. ⧫

MC 3. Find the length of the curve whose parametric equations are $x = \ln\cos t, y = t, 0 \le t \le \pi/4$. a) $\ln(2\sqrt{2}+1)$; b) $\ln(\sqrt{2}+1)$; c) $\ln(2\sqrt{2}+2)$; d) $\ln(\sqrt{2}+1)$; e) $\ln(2\sqrt{3}+1)$; f) $\ln(\sqrt{3}+2)$. ◊

The length of a parametrized curve is

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$$

Applying this formula with $dx/dt = -\sin t/\cos t = -\tan t$ and $dy/dt = 1$ gives

$$L = \int_0^{\pi/4} \sqrt{\tan^2 t + 1}\, dt = \int_0^{\pi/4} \sqrt{\sec^2 t}\, dt = \int_0^{\pi/4} \sec t\, dt.$$

(The last step is valid since $\sec t \ge 0$ between $t = 0$ and $\pi/4$.) From here there are two methods to evaluate this integral.

Method 1: This is my method, and involves dogged substitutions. The substitutions I use are $u = \cos t$, $v = 1 - u^2$, and $w = \sqrt{v}$.

$$\begin{aligned}
L &= \int_0^{\pi/4} \sec t \, dt \\
&= -\int_1^{1/\sqrt{2}} \frac{du}{u\sqrt{1-u^2}} \\
&= \frac{1}{2}\int_0^{1/2} \frac{dv}{(1-v)\sqrt{v}} \\
&= \frac{1}{2}\int_0^{1/\sqrt{2}} \frac{2}{1-w^2}\, dw \\
&= \frac{1}{2}\int_0^{1/\sqrt{2}} \frac{1}{1-w} + \frac{1}{1+w}\, dw \\
&= \frac{1}{2}(-\ln|1-w| + \ln|1+w|)\Big|_0^{1/\sqrt{2}} \\
&= \frac{1}{2}\left(-\ln\left(1-\frac{1}{\sqrt{2}}\right) + \ln\left(1+\frac{1}{\sqrt{2}}\right)\right) \\
&= \frac{1}{2}\left(\ln\frac{(\sqrt{2}+1)/\sqrt{2}}{(\sqrt{2}-1)/\sqrt{2}}\right) \\
&= \frac{1}{2}\ln\frac{(\sqrt{2}+1)(-\sqrt{2}-1)}{-1} \\
&= \ln\sqrt{3+2\sqrt{2}}.
\end{aligned}$$

Now, $(1+\sqrt{2})^2 = 3 + 2\sqrt{2}$, so

$$\begin{aligned}
L &= \ln\sqrt{3+2\sqrt{2}} \\
&= \ln(1+\sqrt{2}).
\end{aligned}$$

Hence, the correct answer is d) $\ln(\sqrt{2}+1)$. ♦

Method 1′: This method rolls the three substitutions of Method 1 into one less obvious substitution $u = \sin t$.

$$\begin{aligned}
L &= \int_0^{\pi/4} \sec t \, dt \\
&= \int_0^{\pi/4} \frac{\cos t}{1-\sin^2 t}\, dt \\
&= \int_0^{1/\sqrt{2}} \frac{du}{1-u^2}.
\end{aligned}$$

From this point forward, Method 1′ is the same as Method 1. ♦

Method 2: I don't think anyone short of a true genius would know this method without seeing it at least once before. It uses the substitution $u = \sec t + \tan t$.

$$\begin{aligned} L &= \int_0^{\pi/4} \sec t \, dt \\ &= \int_0^{\pi/4} \frac{\sec t(\sec t + \tan t)}{\sec t + \tan t} \, dt \\ &= \int_0^{\pi/4} \frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} \, dt \\ &= \int_1^{1+\sqrt{2}} \frac{du}{u} \\ &= (\ln|u|)\Big|_1^{1+\sqrt{2}} \\ &= \ln|1+\sqrt{2}| = \ln(1+\sqrt{2}). \end{aligned}$$

Hence, the correct answer is d) $\ln(\sqrt{2}+1)$. ♦

MC 4. Find the Cartesian equation for the curve given by the polar equation $r \sin\theta + r^2\cos^2\theta + r^2 = 0$. a) $2x + 2x^2 + y^2 = 0$; b) $x^2 + y + y^2 = 0$; c) $2x^2 + y + y^2 = 0$; d) $x^2 + 2y + 2y^2 = 0$; e) $x^2 + x + y^2 = 0$; f) $x + x^2 + 2y^2 = 0$. ♢

Substituting $r \sin\theta = y$, $r\cos\theta = x$, and $r^2 = x^2 + y^2$, we get

$$\begin{aligned} y + x^2 + (x^2 + y^2) &= 0 \\ 2x^2 + y + y^2 &= 0. \end{aligned}$$

Hence, the correct answer is c) $2x^2 + y + y^2 = 0$. ♦

MC 5. Find the area of the region inside the polar curve $r = \sin\theta$.

Method 1: The picture on the exam sheet suggests that this curve is a circle of radius $1/2$ centered at $(x, y) = (0, 1/2)$. Indeed, we can prove this by converting to Cartesian coordinates:

$$\begin{aligned} r &= \sin\theta \\ r^2 &= r\sin\theta \\ x^2 + y^2 &= y \\ x^2 + y^2 - y + \frac{1}{4} &= \frac{1}{4} \\ x^2 + \left(y - \frac{1}{2}\right)^2 &= \left(\frac{1}{2}\right)^2. \end{aligned}$$

The area of a circle of radius r is πr^2, so the area of this circle is $\pi(1/2)^2 = \pi/4$. Hence, the correct answer is e) $\pi/4$. ♦

Method 2: By inspection, one loop of this curve is parametrized by $r = \sin\theta, 0 \le \theta \le \pi$. Then by the polar formula for area, the area within this curve is

$$\begin{aligned}\int_0^\pi \frac{r^2}{2}\,d\theta &= \int_0^\pi \frac{\sin^2\theta}{2}\,d\theta \\ &= \int_0^\pi \frac{1}{2}\cdot\frac{1-\cos 2\theta}{2}\,d\theta \\ &= \int_0^\pi \frac{d\theta}{4} - \frac{1}{4}\int_0^\pi \cos 2\theta\,d\theta \\ &= \frac{\pi}{4} - \frac{1}{8}(\sin 2\theta)\Big|_0^\pi \\ &= \frac{\pi}{4}.\end{aligned}$$

(If you haven't memorized the identity $\sin^2\theta = (1-\cos 2\theta)/2$, you can get it by solving the simultaneous equations $\sin^2\theta + \cos^2\theta = 1, \cos^2\theta - \sin^2\theta = \cos 2\theta$ for $\sin^2\theta$.) Hence, the correct answer is e) $\pi/4$. ♦

MC 6. Find the area of the shaded region inside both of the polar curves $r = 4\sin\theta$ and $r = 4\cos\theta$. ♢

Method 1: By the argument of Method 1 of problem 5, the two curves are circles of radius 2, one centered at the point $P_1 = (x, y) = (2, 0)$ and $P_2 = (x, y) = (0, 2)$. The circles (call the one centered on the x-axis C_1 and the other C_2) meet at the origin O and at the point $Q = (x, y) = (2, 2)$. The given region is the (disjoint union) of the segment OQ of C_1 and the segment OQ of C_2.

The area of the segment OQ of C_1 is the difference between the area of the sector P_1OQ and the area of the triangle P_1OQ. Since the angle between P_1O and P_1Q is $\pi/2$, the area of the sector P_1OQ is

$$\frac{\pi/2}{2\pi}\pi(2)^2 = \pi.$$

By the well-known formula for the area of a triangle, the area of the triangle P_1OQ is $(2)(2)/2 = 2$. Hence, the area of the segment OQ of C_1 is $\pi - 2$. Similarly, the area of the segment OQ of C_2 is $\pi - 2$. Hence, the area of the shaded region is $2(\pi - 2) = 2\pi - 4$, so the correct answer is a) or f) $2\pi - 4$. ♦

Method 2: The part of the region below the line $y = x$ (in polar coordinates,

$\theta = \pi/4$) is defined by $0 \le r \le 4\sin\theta$ and $0 \le \theta \le \pi/4$. Its area is then

$$\begin{aligned}\int_0^{\pi/4} \frac{(4\sin\theta)^2}{2}\,d\theta &= 8\int_0^{\pi/4} \sin^2\theta\,d\theta \\ &= 4\int_0^{\pi/4} 1-\cos 2\theta\,d\theta \\ &= 4(\theta - (\sin 2\theta)/2)\big|_0^{\pi/4} \\ &= 4\Big(\frac{\pi}{4} - \frac{1}{2}\Big) \\ &= \pi - 2.\end{aligned}$$

Likewise, the part of the region above the line $y = x$ is defined by $0 \le r \le 4\cos\theta$ and $\pi/4 \le \theta \le \pi/2$. Its area is then

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{(4\cos\theta)^2}{2}\,d\theta &= 8\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta \\ &= 4\int_{\pi/4}^{\pi/2} 1+\cos 2\theta\,d\theta \\ &= 4(\theta + (\sin 2\theta)/2)\big|_{\pi/4}^{\pi/2} \\ &= 4\Big(\frac{\pi}{2} + 0 - \frac{\pi}{4} - \frac{1}{2}\Big) \\ &= \pi - 2.\end{aligned}$$

Hence, the area of the entire region is $2\pi - 4$, and the correct answer is a) or f) $2\pi - 4$. ♦

MC 7. Find a unit vector in the direction of $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. a) $\mathbf{i} + \mathbf{j} + \mathbf{k}$; b) $\mathbf{i}$; c) $\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$; d) $\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}$; e) $\frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \mathbf{k}$; f) $\mathbf{i} + \mathbf{j}$. ♢

Since

$$|\mathbf{v}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3,$$

the vector $\mathbf{v}/3 = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ is a unit vector in the direction of $\mathbf{v}$. Hence, the correct answer is c) $\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$. ♦

MC 8. For what value of x will the vectors $\mathbf{v} = \langle x, 3\rangle$ and $\mathbf{w} = \langle 4, 5\rangle$ be orthogonal? a) $-4/15$; b) $15/4$; c) $20/3$; d) $-15/4$; e) $3/20$; f) $4/15$. ♢

$\mathbf{v} \perp \mathbf{w}$ when $\mathbf{v}\cdot\mathbf{w} = 0$. Since $\mathbf{v}\cdot\mathbf{w} = x\cdot 4 + 3\cdot 5 = 4x + 15$, we solve $4x + 15 = 0$: $x = -15/4$. Hence, the correct answer is d) $-15/4$. ♦

MC 9. Find the length of the cardioid $r = 1 + \sin\theta$. a) 2π; b) $\pi/2$; c) π; d) 0; e) 8; f) $3\pi/2$; g) 16. ♢

The length of a curve given in polar coordinates is

$$L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta.$$

Applying this formula to the cardioid $r = 1 + \sin\theta$ from $\theta = 0$ to 2π gives

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta. \end{aligned}$$

From here, I see two ways to evaluate this integral.

Method 1:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\cos(\theta - \pi/2)}\, d\theta \\ &= \int_0^{2\pi} 2\sqrt{\frac{1 + \cos(\theta - \pi/2)}{2}}\, d\theta \\ &= \int_0^{2\pi} 2\sqrt{\cos^2(\theta/2 - \pi/4)}\, d\theta \\ &= \int_0^{3\pi/2} 2\cos(\theta/2 - \pi/4)\, d\theta - \int_{3\pi/2}^{2\pi} 2\cos(\theta/2 - \pi/4)\, d\theta \\ &= 4\sin(\theta/2 - \pi/4)\big|_0^{3\pi/2} - 4\sin(\theta/2 - \pi/4)\big|_{3\pi/2}^{2\pi} \\ &= 4(1 - (-2\sqrt{2})) - 4(2\sqrt{2} - 1) \\ &= 8. \end{aligned}$$

Hence, the correct answer is e) 8. ⧫

Method 2: Using the substitution $u = 1 - \sin\theta$ with $du = -\cos\theta\, d\theta$,

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta \\
&= \sqrt{2} \int_0^{2\pi} \sqrt{\frac{1 - \sin^2\theta}{1 - \sin\theta}}\, d\theta \\
&= \sqrt{2} \int_0^{2\pi} \sqrt{\frac{\cos^2\theta}{1 - \sin\theta}}\, d\theta \\
&= \sqrt{2} \int_{-\pi/2}^{3\pi/2} \sqrt{\frac{\cos^2\theta}{1 - \sin\theta}}\, d\theta \\
&= \sqrt{2} \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1 - \sin\theta}}\, d\theta + \sqrt{2} \int_{\pi/2}^{3\pi/2} -\frac{\cos\theta}{\sqrt{1 - \sin\theta}}\, d\theta \\
&= -\sqrt{2} \int_2^0 \frac{du}{\sqrt{u}} + \sqrt{2} \int_0^2 \frac{du}{\sqrt{u}} \\
&= 4\sqrt{2} \int_0^2 \frac{du}{2\sqrt{u}} \\
&= 4\sqrt{2}(\sqrt{u})\Big|_0^2 \\
&= 4\sqrt{2}\sqrt{2} = 8.
\end{aligned}$$

Hence, the correct answer is e) 8. ♦

MC 10. Find a unit vector orthogonal to both $\langle 1, -1, 0\rangle$ and $\langle 1, 2, 3\rangle$. a) $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$; b) $\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$; c) $\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$; d) $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\rangle$; e) $\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$; f) $\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\rangle$. ◊

Let $\mathbf{a} = \langle 1, -1, 0\rangle$ and $\mathbf{b} = \langle 1, 2, 3\rangle$. Then the cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} \times \mathbf{b} = (-1 \cdot 3 - 0 \cdot 2)\mathbf{i} - (1 \cdot 3 - 0 \cdot 1)\mathbf{j} + (1 \cdot 2 - (-1) \cdot 1)\mathbf{k} = -3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}.$$

The magnitude of this cross product is $\sqrt{(-3)^2 + (-3)^2 + 3^2} = 3\sqrt{3}$, so the unit vectors

$$\pm\frac{1}{3\sqrt{3}}\langle -3, -3, 3\rangle = \pm\left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$$

are orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Hence, the correct answer is d) $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\rangle$. ♦

Free 1. Find the area of the surfaces formed by rotating the given curves about the specified axis.

a. [Redacted.]

b. $r = \sqrt{\cos 2\theta}, 0 \le \theta \le \pi/4$ about the directrix (x-axis). ◊

The surface area of a surface of revolution about the x-axis given in polar coordinates is

$$SA = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta.$$

Since $\frac{dr}{d\theta} = -\sin 2\theta / \sqrt{\cos 2\theta}$, applying this formula gives

$$\begin{aligned} SA &= \int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta + \sin^2 2\theta/\cos 2\theta}\, d\theta \\ &= \int_0^{\pi/4} 2\pi \sin\theta \cos^2 2\theta + \sin^2 2\theta\, d\theta \\ &= \int_0^{\pi/4} 2\pi \sin\theta\, d\theta \\ &= 2\pi(-\cos\theta)\Big|_0^{\pi/4} \\ &= 2\pi\Big(-\frac{\sqrt{2}}{2} + 1\Big) \\ &= \pi(2 - \sqrt{2}). \blacklozenge \end{aligned}$$

2. Below are five polar/parametric graphs and a selection of equations. Match each graph with the equation that produces it. ♢

A. (3) $r\cos\theta = 1$. ♦

B. (1) $r = \cos 2\theta$. ♦

C. (4) $r = 1 + \cos\theta$. ♦

D. (2) $r = \theta$. ♦

E. (5) $r = -\sin\theta$. ♦

3.a. Find a vector orthogonal to the plane containing the points $P = (1, 0, -1)$, $Q = (2, 4, 5)$, and $R = (3, 1, 7)$. ♢

We need a vector orthogonal to the vectors $\overrightarrow{PQ} = \langle 1, 4, 6\rangle$ and $\overrightarrow{PR} = \langle 2, 1, 8\rangle$ which are tangent to the plane PQR. To find such a vector, we take the cross product

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4\cdot 8 - 6\cdot 1)\mathbf{i} - (1\cdot 8 - 2\cdot 6)\mathbf{j} + (1\cdot 1 - 4\cdot 2)\mathbf{k} = 26\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}. \blacklozenge$$

b. What is the area of the triangle whose vertices are P, Q, and R? ♢

The area is half the magnitude of the cross product:

$$\begin{aligned} A &= \frac{|\overrightarrow{PQ} \times \overrightarrow{PR}|}{2} \\ &= \frac{\sqrt{26^2 + 4^2 + (-7)^2}}{2} \\ &= \frac{\sqrt{676 + 16 + 49}}{2} \\ &= \frac{\sqrt{741}}{2}. \blacklozenge \end{aligned}$$

4. Show that the vector $\mathbf{v} = \mathbf{b} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ is orthogonal to $\mathbf{a}$. $\lozenge$

We must show that $\mathbf{v} \cdot \mathbf{a} = 0$.

$$\begin{aligned} \mathbf{v} \cdot \mathbf{a} &= \left(\mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}\right) \cdot \mathbf{a} \\ &= \mathbf{b} \cdot \mathbf{a} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} \cdot \mathbf{a} \\ &= \mathbf{a} \cdot \mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right)|\mathbf{a}|^2 \\ &= \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} \\ &= 0. \blacklozenge \end{aligned}$$