No homework for the next week.
Remember, the Second exam will be on Wednesday, March 25 during the regular class time.

March 22, 2015

1 Problem 1. True or False.

Solution:

1. True.
2. False.
3. False.
4. False.
5. True.

2 Problem 2. A subspace $V$ of $\mathbb{R}^3$ is spanned by the columns of

\[
A = \begin{pmatrix}
1 & 1 \\
-1 & 0 \\
1 & 1
\end{pmatrix}
\]

(a) Apply the Gram-Schmidt process to find two orthonormal vectors $u_1$ and $u_2$ which also span $V$.

Solution:

By the process of Gram-Schmidt, we just follow the steps:
Firstly, let
\[ v_1 = c_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \text{normalizing } u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \]
and
\[ v_2 = c_2 - \frac{c_2^T}{v_1^T v_1} v_1 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \text{normalizing } u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \]

(b) Find an orthogonal matrix \( Q \) so that \( QQ^T \) is the matrix which orthogonally projects vectors onto \( V \).

**Solution:**
The general approach \( P = A(A^T A)^{-1} A^T \).
Take
\[ Q = (u_1 \ u_2) = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \]
since \( Q^T Q = I_{2\times2} \) then \( P = QQ^T = u_1 u_1^T + u_2 u_2^T \) is the required projection.

(c) Find the best possible solution to the linear system
\[ Q = (x \ y) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \]

**Solution:**
Minimizing the sum of squared errors \( \|Qx - b\|^2 \) with respect to \( x \) yields
\[ Q^T Qx = Q^T b \]
But \( Q^T Q = I_{2\times2} \) by ortho-normality. So we find
\[ x = Q^T b = \begin{pmatrix} 2/\sqrt{3} \\ 5/\sqrt{6} \end{pmatrix} \]

### 3 Problem 3. Given

\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \]

(a) Find the matrix \( P \) which projects onto the column space of \( A \).

**Solution:**
Note that the projection on the plane spanned by \( (1, 0, 0)^T \) and \( (1, 1, 0)^T \) is the same as the projection on the plane spanned by \( (1, 0, 0)^T \) and \( (0, 1, 0)^T \), so it is simply
\[ P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

We can also use the general approach \( P = A(A^T A)^{-1} A^T \), which is longer but gives the same answer.

(b) Compute the projection \( p \) of \( b \) onto this column space.

**Solution:**

\[ p = Pb = (2, 3, 0)^T \]

(c) Find the error \( e = b - p \) and show that it lies in the left nullspace of \( A \).

**Solution:**

Apparently \( e = b - p = (0, 0, 4)^T \), and

\[ A^T e = 0 \]

which means \( e \) is in the left null space of \( A \).

### 4 Problem 4. Calculate the determinant of the matrix \( A \).

**Solution:**

We calculate step by step:

\[
\begin{align*}
\det A &= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 2 & 6 & 4 & 8 \\ 3 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 2 & -2 & 0 \\ 0 & -5 & -8 & -10 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & 0 \\ 0 & -4 & -8 & -12 \\ 0 & -5 & -8 & -10 \end{vmatrix} \\
&= -2 \begin{vmatrix} -12 & -12 \\ -13 & -10 \end{vmatrix} = -2(120 - 156) = 72
\end{align*}
\]

### 5 Problem 5. If you know all 16 cofactors of a 4 by 4 invertible matrix \( A \), how would you find its determinant \( \det(A) \)?

**Solution:**

Since \( \det(A) \neq 0 \), by Cramer’s Rule, we have (the Cofactor formula):

\[
A^{-1} = \frac{C^T}{\det(A)}.
\]
Multiply both sides by $A$:

$$AA^{-1} = \frac{AC^T}{\det(A)} \implies I = \frac{AC^T}{\det(A)} \implies det(A)I = AC^T$$

Take the determinant in the last equality:

$$det(A)^4 = det(A)det(C^T) \implies det(A)^3 = det(C^T).$$

Finally,

$$det(A) = det(C^T)^{1/3}.$$