Practice exam - will not be collected

1. (20 points) For each of the following statements, mark whether they are true or false. (5 problems 4 points each).

   (a) For a matrix A, in the Singular Value Decomposition (SVD) $A = U\Sigma V^T$ the factor U is an orthogonal matrix. **ANS: T**

   (T) **(F)**

   (b) If $\det(A^2) = 1$ then A's eigenvalues must all be 1 or -1. **ANS: F**

   (T) **(F)**

   (c) For a symmetric positive definite matrix A all pivots are positive. **ANS:T**

   (T) **(F)**

   (d) A 5×5 matrix B has eigenvalue $\lambda = 3$ of algebraic multiplicity 5 and geometric multiplicity 1 and the matrix B is therefore diagonalizable. **ANS: F**

   (T) **(F)**

   (e) All the eigenvalues of a real symmetric matrix are real. **ANS: T**

   (T) **(F)**

2. (20 points) Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Calculate the eigenvalues of A.

(b) Calculate the eigenvectors of A and write the decomposition $A = SDS^{-1}$ with the diagonal matrix D.

(c) Write an expression for A^k for any positive integer k.

(d) Write an expression for e^A.

SOLN: We calculate

$$det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 0 & \lambda & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & -\lambda & -\lambda \\ 0 & \lambda & 0 \end{vmatrix} = -\lambda(\lambda+1)(\lambda-2)$$

a) The eigenvalues are $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 2$.

For the eigenvectors, we solve the systems $(A - \lambda_s I)e_s = 0, \quad s = 1, 2, 3$.

Let $e_s = [x_s, y_s, z_s]$. We have to solve the following systems:

$$\begin{array}{ccc} x_1 + y_1 + z_1 = 0, & 2x_2 + y_2 + z_2 = 0, & -x_3 + y_3 + z_3 = 0 \\ x_1 = 0, & x_2 + y_2 = 0, & x_3 - 2y_3 = 0 \\ x_1 = 0, & x_2 + z = 0, & x_3 - 2z_3 = 0 \end{array}$$

The matrix A is symmetric with three different eigenvalues. Therefore the three eigenvectors are orthogonal. We normalize each eigenvector to get an orthogonal matrix $S = Q$.

Solving the systems above, we select three orthonormal eigenvectors:

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad e_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad e_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

The diagonal matrix D equals

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The matrix S is:

$$S = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

and it is an orthogonal matrix, $S^{-1} = S^T$.

b) The factorization of A is:

$$A = SDS^{-1} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

c) Using $A = SDS^{-1}$ we have $A^k = SD^kS^{-1}$ and

$$D^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & 2^k \end{pmatrix}.$$

d) Using the expansion of e^A and c) we calculate

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots = S(I + \frac{D}{1!} + \frac{D^2}{2!} + \ldots)S^T.$$

Using the expansion of e^t we obtain

$$e^A = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} e^0 & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

3. (20 points) Find the Singular Value Decomposition (SVD) of the matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(a) Find the eigenvalues and unit length eigenvectors for $A^T A$ and AA^T. (What is the rank of A?)

(b) Calculate the three matrices U_r, Σ_r, V_r in the reduced SVD and observe $AV_r = U_r\Sigma_r$.

(c) Calculate the three matricies U, Σ, and V^T in the SVD and observe that $A = U\Sigma V^T$. Please explain clearly how you obtain these matrices.

SOLN:

a) Calculate:

$$A^T A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

We have $det(A^T A - \lambda I) = \lambda^2 - 4\lambda + 3$ with eigenvalues $\lambda_1 = 3, \quad \lambda_2 = 1$.

The rank of A is two.

Solving $(A^T A - 3I)v_1 = 0$ and $(A^T A - I)v_2 = 0$ we obtain

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We have

$$AA^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

We have $det(AA^T - \lambda I) = (\lambda - 1)(-\lambda^2 + 3\lambda)$ with eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0$.

Solving $(A^TA-3I)u_1=0$, $(A^TA-I)u_2=0$ and $A^TAu_3=0$ we obtain

$$u_1=\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix},\quad u_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix},u_3=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

b) Observe $u_1=\frac{1}{\sqrt{3}}Av_1,\quad u_2=Av_2$, and $u_3\in N(A^T)$.

Hence $V_r=[v_1v_2], U_r=[u_1u_2]$ with the expressions

$$V_r=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{pmatrix},\quad U_r=\begin{pmatrix}\frac{1}{\sqrt{6}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}\\\frac{2}{\sqrt{6}}&0\end{pmatrix}$$

and

$$AV_r=U_r\Sigma_r,\quad \Sigma_r=\begin{pmatrix}\sqrt{3}&0\\0&1\end{pmatrix}$$

c) Since $rank(A)=2$ and A is 3 by 2 matrix, $V=V_r$. To obtain U we need to add one more column to U_r from $N(A^T)$ of norm one, $U=[u_1u_2u_3]$ and to get Σ we need one more row of zeros to be added to Σ_r.

The SVD is:

$$A=U\Sigma V^T=\begin{pmatrix}\frac{1}{\sqrt{6}}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\\\frac{2}{\sqrt{6}}&0&-\frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}\sqrt{3}&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{pmatrix}$$

4. (20 points) Calculate the Jordan form of the following matrix:

$$B = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

(a) Calculate the eigenvalues of B and show $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 1$.

(b) Find the algebraic and geometric multiplicities of both eigenvalues.

(c) Find the Jordan form J of B.

(d) Find an invertible matrix S so that $B = SJS^{-1}$. Be sure to clearly explain how you obtained the columns of this matrix.

SOLN The problem was solved in class, see your class notes.

5. (20 points) Consider the matrix:

$$B = \begin{pmatrix} -1 & -8 & 4 \\ -8 & -1 & -4 \\ 4 & -4 & -7 \end{pmatrix}.$$

(a) Calculate the eigenvalues of B and show $\lambda_1 = \lambda_2 = -9$, $\lambda_3 = 9$.

(b) Find the algebraic and geometric multiplicities of both eigenvalues.

(c) Find the diagonal form Λ of B

(d) Find an orthogonal matrix Q so that $B = Q\Lambda Q^T$. Be sure to clearly explain how you obtained the columns of this matrix.

SOLN: The problem was solved in class, see your class notes.