Review for Midterm 1

#57, 5.2. How many ways to place 8 identical black pieces and 8 identical white pieces on an 8x8 chessboard?

$$\binom{64}{8,8} = \binom{64}{8}\binom{56}{8} = \frac{64!}{8!\,8!\,48!} = \frac{64\cdot 63\cdots 49}{8!\,8!}$$

#59, 5.3. What is the probability that two (or more) people in a random group of 25 people have a common birthday?

$$1-\left(1\cdot\frac{364}{365}\,\frac{363}{365}\cdots\frac{341}{365}\right) \doteq .5687$$

#3, 5.3 (a) How many #'s, 8 digits, can be formed from 3's, 5's, and 7's

3^8

(b) What fraction of the #'s from (a) have three 3's, two 5's, and three 7's?

$$\frac{\binom{8}{3,2,3}}{3^8} = \frac{8!}{3^8\,3!\,2!\,3!}$$

#8 5.4 How many ways to arrange 12 identical apples and 5 different oranges in a row so that no two oranges appear side by side?

Start by placing oranges in a row in $5!$ ways

$$\_\ O_1\ \_\ O_2\ \_\ O_3\ \_\ O_4\ \_\ O_5\ \_$$

There are 6 places to put the apples, and

we need at least one apple in places 2, 3, 4, & 5.
Let x_i = # apples place i

$$x_1 + x_2 + \cdots + x_6 = 12 \qquad x_1, x_6 \geq 0 \quad x_2, x_3, x_4, x_5 \geq 1$$

= # weak composition of 8 into 6 parts

$$= \binom{8+5}{5} \qquad \text{Total: } \binom{13}{5} 5! = 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$$

#10 5.4 How many ways are there to arrange the 26 letters of the alphabet so that no pair of vowels appear consecutively?

Vowels: a, e, i, o, u

Start by arranging vowels in 5! ways, as in

_ e _ o _ a _ i _ u _

6 spots to place remaining 21 letters, at least one in spots 2, 3, 4, & 5. Arrange 21 letters in order in 21! ways. Then distribute in $\binom{17+5}{5}$ ways

Total $5!\, 21! \binom{22}{5} = 21!\, 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18$

ex. Prove

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1} \qquad n \geq r$$

by induction on n

Base Case $n = r$

$$\binom{r}{r} = \binom{r+1}{r+1} \quad \text{or} \quad 1 = 1 \quad \checkmark$$

Inductive Step

$$\binom{r}{r} + \cdots + \binom{n}{r} = \underbrace{\binom{r}{r} + \cdots + \binom{n-1}{r}} + \binom{n}{r}$$

$$= \binom{n}{r+1} + \binom{n}{r} \qquad \text{by induction}$$

$$= \binom{n+1}{r+1} \qquad \checkmark$$

ex

Prove $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

using generating functions

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = (1+x)^n (1+x)^n \Big|_{x^n}$$

$$= (1+x)^{2n} \Big|_{x^n} = \binom{2n}{n}$$

ex #17 6.1 Find a gen. func. for the # of selections of p sticks of chewing gum chosen from 8 different flavors if each flavor comes in packets of 5 sticks

$$(1 + x^5 + x^{10} + x^{15} + \dots)^8$$

ex. #28, 6.2 If a coin is flipped 25 times with 8 tails occurring, what is the probability that no run of six (or more) consecutive heads occurs?

sequences with no run of 6 (or more)

$$\_\ T_1\ \_\ T_2\ \_\ T_3\ \_\ T_4\ \_\ T_5\ \_\ T_6\ \_\ T_7\ \_\ T_8\ \_$$

9 boxes between Tails to put 17 Heads in, no box with more than 5 Heads,

$$= (1 + x + x^2 + x^3 + x^4 + x^5)^9 \Big|_{x^{17}} = \left(\frac{1-x^6}{1-x}\right)^9 \Big|_{x^{17}}$$

$$= \left(1 - \binom{9}{1}x^6 + \binom{9}{2}x^{12} - \binom{9}{3}x^{18} \dots\right) \sum_{n=0}^{\infty} \binom{n+8}{n} x^n \Big|_{x^{17}}$$

$$= \binom{17+8}{17} - 9 \cdot \binom{11+8}{11} + \binom{9}{2}\binom{5+8}{5}$$

$$\text{Prob.} = \frac{\binom{25}{17} - 9\cdot\binom{19}{11} + 36\cdot\binom{13}{5}}{\binom{25}{8}}$$

ex. #7 (b) Show using gen. funcs that the # of partitions of n into distinct parts equals the # of partitions of n into odd parts

$$\sum_{n=0}^{\infty} x^n (\text{\# partitions of } n \text{ into distinct parts})$$

$$= (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$

$$= \frac{1-x^2}{1-x}\,\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\,\frac{1-x^8}{1-x^4}\cdots$$

$$= \frac{1}{1-x}\,\frac{1}{1-x^3}\,\frac{1}{1-x^5}\,\frac{1}{1-x^7}\cdots$$

$$= \sum_{n=0}^{\infty} x^n (\text{\# partitions of } n \text{ into odd parts})$$

ex. #9 6.4 How many 10-letter words are there in which each of the letters e, n, r, s occur

(a) At most once?

(b) At least once?

(a) $$10!\left[(1+x)^4\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^{22}\right]\Big|_{x^{10}}$$

$$= 10!\,(1+x)^4 e^{22x}\Big|_{x^{10}}$$

$$= 10!\sum_{k=0}^{4}\binom{4}{k} e^{22x}\Big|_{x^{10-k}} = \boxed{10!\sum_{k=0}^{4}\binom{4}{k}\frac{22^{10-k}}{(10-k)!}}$$

(b) $$10!\left[\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^4 e^{22x}\right]\Big|_{x^{10}}$$

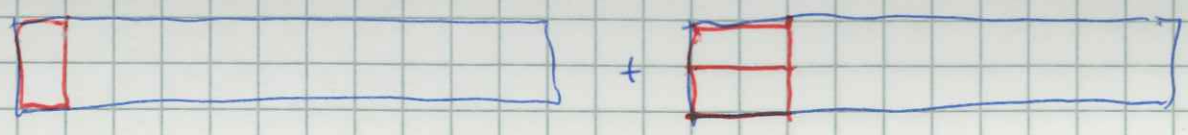
$$= 10!\left[(e^x-1)^4 e^{22x}\right]\Big|_{x^{10}}$$

$$= 10!\sum_{k=0}^{4}\binom{4}{k} e^{kx}(-1)^{4-k} e^{22x}\Big|_{x^{10}}$$

$$= 10!\sum_{k=0}^{4}(-1)^{4-k} e^{(22+k)x}\Big|_{x^{10}} = 10!\sum_{k=0}^{4}(-1)^{4-k}\frac{(22+k)^{10}}{10!}$$

$$= \boxed{\sum_{k=0}^{4}(-1)^{4-k}(22+k)^{10}}$$

#9 7.1 Find a recurrence for the # of ways to arrange n dominoes to fill a 2xn checkerboard



$a_n = a_{n-1} + a_{n-2}$ $\quad a_1 = 1 \quad a_2 = 2$

Fibonacci #'s