#57, 5.2. How many ways to place 8 identical black pieces and 8 identical white pieces on an $n^2$ chessboard?

$$\binom{64}{8, 8} = \frac{64!}{8!8!} = \frac{64 \cdot 63 - 49}{8!8!}$$

#59, 5.1. What is the probability that two (or more) people in a random group of 25 people have a common birthday?

$$1 - \left( 1 - \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{311}{365} \right) = 2.5687$$

#53, 5.3 (a) How many #15, 8 digit, can be formed from $3^{15}$, $5^{15}$, and $7^{15}$?

(b) What fraction of the #15 from (a) have three $3^{15}$, two $5^{15}$, and three $7^{15}$?

$$\binom{8}{3, 2, 3} = \frac{8!}{3!2!3!} = 38$$

#8 5.4 How many ways to arrange 10 identical apples and 5 different oranges in a row so that no two orange appear side by side?

Start by placing orange in a row in 5! ways:

- \(O_1\) - \(O_2\) - \(O_3\) - \(O_4\) - \(O_5\) -

There are 6 places to put the apple, and
we need at least one apple in places 3, 3, 3, 5.

Let \( X_i \) = # apples place i.

\[ X_1 + X_2 + \ldots + X_6 = 12 \]

\[ X_1, X_2 \geq 3 \]

\[ X_3, X_4, X_5, X_6 \geq 3 \]

= # weak compositions of 8 into \( \geq 2 \) parts

\[ \binom{13}{5} \]

Total: \( \left( \begin{array}{c} 13 \\ 5 \end{array} \right) \times 5^1 = 13 \cdot 13 \cdot 11 \cdot 10 \cdot 9 \]

#10 5.4 How many ways are there to arrange the 26 letters of the alphabet so that no pair of vowels appear consecutively?

Vowels: a, e, i, o, u

Start by arranging vowels in \( 5! \) ways, a i e o u

_ e o a i u _

6 spots to place remaining 21 letters, at least one in spots 2, 3, 4, 5. Arrange 21 letters in order in \( 21! \) ways. Then distribute in \( \left( \begin{array}{c} 17+5 \\ 5 \end{array} \right) \) ways

Total \( 5! \cdot 21! \cdot \left( \begin{array}{c} 22 \\ 5 \end{array} \right) = 21! \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \)
Ex. Now
\[ \binom{n}{r} + \binom{n}{r+1} + \cdots + \binom{n}{n} = \binom{n+1}{r+1} \quad \text{for } n \geq r \]

by induction on \( n \).

**Base Case** \( n = r \)
\[ \binom{r}{r} = \binom{r+1}{r+1} \quad \text{and} \quad 1 = 1 \quad \checkmark \]

**Inductive Step**
\[ \binom{n}{r} + \cdots + \binom{n}{n} = \underbrace{\binom{n}{r} + \cdots + \binom{n}{r-1} + \binom{n}{n}}_{\text{by induction}} \]
\[ = \binom{n+1}{r+1} \quad \checkmark \]

Prove \( \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \)

using generating functions
\[ \sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{k}{k} \binom{n}{n-k} = (1+x)^n (1+x)^n \bigg/_{x^n} \]
\[ = (1+x)^{2n} \bigg/_{x^n} = \binom{2n}{n} \]
ex #17 6.1 Find a generating function for the number of selections of $p$ sticks of chewing gum chosen from 8 different flavors if each flavor comes in packets of 5 sticks:
\[(1 + x^5 + x^{10} + x^{15} + \ldots)^8 \]

ex. #28, 6.2 A coin is flipped 25 times with 8 tails occurring. What is the probability that no run of six (or more) consecutive heads occurs? 

# sequences with no run of 6 (or more)

\[
\begin{array}{cccccccc}
T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\
\end{array}
\]

9 boxes between Tails to put 17 Heads in, no box with more than 5 Heads,

\[
\left(1 + x + x^2 + x^3 + x^4 + x^5\right)^9 \]

\[
\left(\frac{1-x^6}{1-x}\right)^9
\]

\[
\left(1 - \binom{9}{1}x^6 + \binom{9}{2}x^{12} - \binom{9}{3}x^{18} + \ldots\right)
\]

\[
\sum_{n=0}^{\infty} \binom{n+8}{n} x^n
\]

\[
(17+8) - 9 \binom{11+8}{11} + \binom{9}{2} \binom{5+8}{5}
\]
Problem: \( (\frac{25}{7}) - 9 \cdot \left( \frac{19}{11} \right) + 36 \cdot \left( \frac{13}{5} \right) \)

\( \frac{2^5}{8} \)

Example #7 (a) Show using generating functions that the number of partitions of \( n \) into distinct parts equals the number of partitions of \( n \) into odd parts.

\[
\sum_{n=0}^{\infty} x^n \left( \text{# partitions of } n \text{ into distinct parts} \right)
\]

\[
= \frac{1}{1-x} \left( \frac{1}{1-x^2} \right) \left( \frac{1}{1-x^3} \right) \left( \frac{1}{1-x^4} \right) \cdots
\]

\[
= \frac{1-x^2}{1-x} \frac{1-x^4}{1-x^2} \frac{1-x^6}{1-x^3} \frac{1-x^8}{1-x^4} \cdots
\]

\[
= \frac{1}{1-x} \frac{1}{1-x^3} \frac{1}{1-x^5} \frac{1}{1-x^7} \cdots
\]

\[
= \sum_{n=0}^{\infty} x^n \left( \text{# partitions of } n \text{ into odd parts} \right)
\]

Example #9 6.4 How many 10-letter words are there in which each of the letters e, n, r, s occur

(a) At most once?

(b) At least once?
(a) \[ 10! \left[ (1 + x)^4 \left( 1 + \frac{x^2}{3!} + \frac{x^3}{3!} + \cdots \right)^2 \right] \bigg|_{x^{10}} \]

\[ = 10! \left( 1 + x \right)^{4} e^{2\beta x} \bigg|_{x^{10}} \]

\[ = 10! \sum_{k=0}^{10} \binom{4}{k} \frac{(2\beta)^k}{k!} e^{2\beta x} \bigg|_{x^{10}} = \left[ 10! \sum_{k=0}^{10} \binom{4}{k} \frac{(2\beta)^k}{k!} \right] \bigg|_{x^{10}} \]

(b) \[ 10! \left[ \left( x + \frac{x^2}{3!} + \frac{x^3}{3!} + \cdots \right)^4 e^{2\beta x} \right] \bigg|_{x^{10}} \]

\[ = 10! \left[ (e^x - 1)^4 e^{2\beta x} \right] \bigg|_{x^{10}} \]

\[ = 10! \sum_{k=0}^{10} \binom{4}{k} e^{2\beta x} \bigg|_{x^{10}} \]

\[ = 10! \sum_{k=0}^{10} \binom{4}{k} \frac{(2\beta)^k}{k!} e^{2\beta x} \bigg|_{x^{10}} = 0! \sum_{k=0}^{10} \frac{(-1)^{4-k} (2\beta)^k}{k!} \]

\[ = \sum_{k=0}^{10} (-1)^{4-k} (2\beta + k)^{10} \]
# 9.1

Find a recurrence for the # of ways to arrange n dominoes to fill a $2 \times n$ checkerboard.

\[ a_n = a_{n-1} + a_{n-2} \]

- $a_1 = 1$
- $a_2 = 2$

Fibonacci #15