Chapter 8 - Inclusion-Exclusion

Venn Diagrams

If \( A \) is a set, \( \bar{A} \) is all things not in \( A \).

\[ N(A) = |A| = \#A = \# \text{ of elements in } A \]

\[ N = N(U) = \text{ total \# of elements in a universe } U \]

\[ N(A) = N - N(A) \]

\[ N(\bar{F} \cap \bar{L}) = N(F \cup L) - N(F \cap L) \]

De Morgan's Law: \( \overline{F \cap L} = \bar{F} \cup \bar{L} \)

So \( N(\bar{F} \cap \bar{L}) = N(\bar{F} \cup \bar{L}) = N - N(F \cup L) \)

So \( N(F \cap L) = N - (N(F) + N(L) - N(F \cap L)) = N - N(F) - N(L) + N(F \cap L) \)
ex. How many arrangements of the digits 0, 1, 2, 9 are there in which the 1st digit is larger than one and the last digit smaller than 8?

\[
\text{N(Full)} = \text{N(Full)}^{\text{1st} \text{ larger}} \cdot \text{N(Full)}^{\text{last} < 8} - \left( \text{N(Full)}^{\text{1st} > 1 \text{ or last} < 8} \right)
\]

\[
= 8 \cdot 9! + 8 \cdot 9! - \left( \text{N(Full)}^{\text{1st} > 1 \text{ or last} < 8} \right)
\]

\[
\text{N(Full)}^{\text{1st} > 1 \text{ or last} < 8} = \text{total #} - \left( \text{N(Full)}^{\text{1st} \leq 1 \text{ and last} \geq 8} \right)
\]

\[
= 10! - \left[ 2 \cdot 2 \cdot 8! \right]
\]

Final Answer: \[8! \left( 2 \cdot 9! - 10! - 4 \cdot 8! \right) = 8! \left( 16 \cdot 9 - 90 + 4 \right) = 8! \left( 58 \right)\]
\[ N(F \cup L \cup G) = N(F) + N(G) + N(L) - N(F \cap L) - N(F \cap G) - N(L \cap G) + N(F \cap L \cap G) \]

\[ N(F \cap L \cap G) = N - N(F) - N(L) - N(G) + N(F \cup L) + N(F \cup G) + N(L \cup G) - N(F \cup L \cup G) \]

Ex. How many \( n \)-digit ternary \((0, 1, 2)\) sequences are there with at least one 0, at least one 1, and at least one 2? How many \( n \)-digit ternary sequences with at least one \( 0 \) or \( 1 \) or \( 2 \) (missing digit)?

\[ F = \text{set with no } 0^3, \quad G = \text{set with no } 1^3, \quad L = \text{set with no } 2^3 \]

\[ N(F \cap L \cap G) = N - N(F) - N(L) - N(G) + N(F \cup L) + N(F \cup G) + N(L \cup G) - N(F \cup L \cup G) \]

\[ = 3^n - 2^n - 2^n - 2^n - 1 + 1 + 1 = 3^n - 3 - 2^n + 3 \]
with at least one void

\[ N(F \cup G \cup L) = N(F) + N(G) + N(L) - N(F \cap G) - N(F \cap L) - N(G \cap L) + N(F \cap G \cap L) \]

\[ = 2^n + 2^n + 2^n - 1 - 1 - 1 + 0 = 3 \cdot 2^n - 3 \]

Ex: Suppose we 100 students in a school + 40 students taking each language French, Latin, and German. 20 students are taking only French, 20 only Latin, 15 only German. 10 are taking French and Latin. How many are taking all 3?

No Language:

F = French Students, G = German Students
L = Latin Students

\[ N = 100 \]
\[ N(F) = N(L) = N(G) = 40 \]

\[ N(F \cap G \cap L) = 20 = N_1 \]
\[ N(L \cap F \cap G) = 30 = N_5 \]
\[ N(G \cap F \cap L) = 15 = N_7 \]
\[ N(F \cap L) = 10 = N_2 + N_3 \]

\[ N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 = 40 \]
\[ N_2 + N_3 = 10 \]
\[ N_5 = 20 \]
\[ N_7 = 15 \]

\[ N_2 + N_3 + N_5 + N_6 = 40 \]
\[ N_5 = 20 \]
\[ N_2 + N_3 = 10 \] => \( N_6 = 10 \)
\[ N_5 + N_4 + N_6 + N_7 = 40 \]
\[ N_7 = 15 \] \[ N_4 + N_6 + N_7 = 40 \] \[ N_3 = 5 \]
So 5 students are taking all 3.

\[ N_1 + N_3 = 10, \quad N_3 = 5 \Rightarrow N_2 = 5 \]

\[ \Rightarrow N - (N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7) \]
\[ = \# \text{ taking no language} \]
\[ = 100 - (20 + 5 + 5 + 10 + 20 + 10 + 5) \]
\[ = 100 - 85 = 15 \Rightarrow N_b \]

ex. #3. How many \( n \)-digit ternary sequences are there in which at least one pair of consecutive digits are the same?

\[ \# \text{ of Ternary} = \text{total} - (\# \text{ no consecutive digits are the same}) \]
\[ = 3^n - 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \]
\[ \Rightarrow 3^n - 3 \cdot 2^{n-1} \]

#7. If you pick a random integer between 1 and 1,000, what is the probability that it is either divisible by 7 or even (or both)?

\[ A = \text{set divisible by 7} \quad B = \text{set even} \]
\[ N(A \cup B) = N(A) + N(B) - N(A \cap B) \]
\[ = \left\lceil \frac{1000}{7} \right\rceil + \left\lceil \frac{1000}{2} \right\rceil - \left\lceil \frac{1000}{14} \right\rceil = 142 + 500 - 71 = 571 \]
ex. #27. How many arrangements are there of
TAMELY with either T before A, or A before M, or M before E ? (by before we mean
anywhere before
F = set with T before A
G = " A before M
H = set with M before E

\[ N(F \cup G \cup H) = N(F) + N(G) + N(H) - N(F \cap G) - N(F \cap H) - N(G \cap H) + N(F \cap G \cap H) \]

\[ = \binom{3}{1} \cdot \frac{2}{3}! \cdot \frac{1}{3}! - \binom{3}{1} \cdot \frac{2}{3}! \cdot \frac{1}{3}! - \binom{3}{1} \cdot \frac{2}{3}! \cdot \frac{1}{3}! + \frac{1}{3}! \]

 satisfying the
condition T + A \text{ letter chosen} \quad A + M \]

ex. #18. How many numbers between 1 and 280 are
relatively prime to 280?

280 = 4 \cdot 70 = 8 \cdot 5 \cdot 7

\#rel. prime = total # - (multiples of 2) - (multiples of 5)
- (multiples of 7) + (multiples of 10) + (multiples of 14)
+ (multiples of 35) - (multiples of 70) = 280 - 140 - 56 - 40 + 28 + 20 + 8 - 4

= 280 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) = 280 \cdot \frac{1}{2} \cdot \frac{4}{5} \cdot \frac{6}{7} = 4 \cdot 24 = 96

More generally, \( \phi(n) = \# \text{ relatively prime to } n \text{ and between 1 and } n \left( \prod_{p \mid n} \left(1 - \frac{1}{p}\right) \right) \) (Euler's totient function)
\[ N(\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}) = N - S_1 + S_2 - S_3 + \ldots + (-1)^n S_n \]

where \( S_k \) is the sum of \( N(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) \), over all \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \)

In this Venn diagram, you can't get all possible elements of each \( S_k \) using circles, but you can using more general shapes.
Also,
\[ N(A \cup A_3 \cup \ldots \cup A_n) = S_1 - S_2 + S_3 - \ldots + (-1)^{n-1}S_n \]

How many ways to select a 6-card hand from a regular 52-card deck such that the hand contains at least one card in each suit?

Let \( S \) = set of hands with at least one card in each suit.

- \( D \) = diamond
- \( H \) = heart
- \( C \) = club
- \( S \) = spade

\[ N(S \cup D \cup H \cup C) = N - S_1 + S_2 - S_3 + S_4 = \binom{52}{6} - \binom{39}{6} + \binom{36}{6} - \binom{26}{6} + \binom{13}{6} \]

\[ S_1 = N(S^6) + N(D^6) + N(H^6) + N(C^6) = \binom{52}{6} \]

\[ S_2 = N(S^5 \cap D^6) + \ldots + N(D^5 \cap C^6) = \binom{39}{6} \]

\[ S_3 = N(S^5 \cap D^5 \cap H^6) + N(D^5 \cap H^5 \cap C^6) = \binom{26}{6} \]

\[ S_4 = \binom{13}{6} \]

\[ N = \binom{52}{6} \]

\[ S_4 = 0 \]
ex. How many integer solutions to
\[ x_1 + x_2 + \cdots + x_6 = 20 \quad 0 \leq x_i \leq 8 \]

For each solution \( \sum_{i=1}^{6} x_i = 20 \) \( x_i \geq 9 \)
other \( x_j = 0 \)

Let \( A_i \) be the number of solutions to \( x_1 + \cdots + x_6 = 20 \) \( x_i \geq 9 \)
other \( x_j = 0 \)

We want \( N(A_1 \cap \overline{A_2} \cap \cdots \cap \overline{A_6}) \)

Now \( N(A_1 \cap \cdots \cap A_6) = N - 5_1 + 5_2 - 5_3 + 5_4 - 5_5 + 5_6 \)

Note that if at least three of the \( x_i \) are \( \geq 9 \) then
The sum is \( \geq 27 \), so \( 5_3 = 5_4 = 5_5 = 5_6 = 0 \)

\[ N = \binom{20+5}{5} \quad 5_1 = \binom{6}{5} \binom{11+5}{5} \quad 5_2 = \binom{6}{2} \binom{3+5}{5} \]

so answer is
\[ \binom{25}{5} - 6 \cdot \binom{16}{5} + 15 \cdot \binom{7}{5} \]

ex. What is the probability that if \( n \) people randomly reach into a dark closet to retrieve their hats, no person will retrieve their own hat? A derangement of the numbers 1 through \( n \) is a permutation \( \sigma \) of 1 through \( n \) where \( \sigma(i) \neq i \) for \( 1 \leq i \leq n \)
What is the probability that a random permutation of \(1, 2, \ldots, n\) is a derangement (i.e., has no fixed pt's)?

Let \(A_i\) - set of \(i\) with \(0_i = i\), so \(N(A_i) = (n-1)!\)

\[N(A_1 \setminus A_2) = (n-2)!, \quad N(A_1 \setminus A_2 \setminus A_3) = (n-3)!\], etc.

\[
\sum_{1 \leq i < j \leq n} \left\{ N(A_1 \setminus A_2 \setminus \ldots \setminus A_k) = \sum_{1 \leq i < j < k \leq n} (-1)^{k-1} S_k \right. \\
= n! - \binom{n}{2} (n-1)! + \binom{n}{3} (n-2)! - \binom{n}{4} (n-3)! + \cdots + \binom{n}{n} (n-n)! + \cdots + (-1)^n n! \\
= n! \left[ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots + \frac{(-1)^k}{k!} + \cdots + \frac{(-1)^n}{n!} \right] \\
\]

So probability is \(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \frac{(-1)^n}{n!} \approx \frac{1}{e}\)

(Taylor's series for \(1/e\))

Ex. 3: What is the prob. that a 10-card hand from a standard 52-card deck has at least one 4 of a kind?

Let \(A_i\) - set of 10 card hands with four cards of

Face value \(i\) (cases face value 1, ... kings face value 13)

\[1 \leq i \leq 13\]. Then

\[
S_i - S_2 = (13) \binom{4}{2} - \binom{13}{2} \binom{4}{2} \\
\]

prob. = \[
\frac{(13) \binom{4}{2} - \binom{13}{2} \binom{4}{2}}{\binom{52}{10}}
\]
ex. #13 How many ways for a child to take 0 pieces of candy if the child does not take exactly 2 pieces of candy of any type, when there are 4 types of candy?

Let $A_i$ = set of ways to take 10 pieces of candy with exactly $i$ pieces of type $i$ for $i \leq 4$.

Clearly $N(A_i)$ = # solutions to $x_1 + x_2 + x_3 + x_4 = 10$

$x_i = 2$, other $x_i \geq 0$, so $N(A_i)$ = # solutions to $x_2 + x_3 + x_4 = 8$

$N(A_i) = \binom{8+2}{2}$

$N(A_1 \cup A_2) = \# solutions to x_3 + x_4 = 6$

$N(A_1 \cup A_2) = \binom{6+1}{1}$

$N(A_1 \cup A_2 \cup A_3) = 1$

$N = \# solutions to $x_1 + x_2 + x_3 + x_4 = 10$

$N = \binom{10+2}{2}$

$N(A_1 \cup \overline{A_2} \cup \overline{A_3} \cup \overline{A_4}) = N - S_1 + S_2 - S_3 + S_4$

$= \binom{13}{3} - 4 \cdot \binom{10}{2} + 6 \cdot 7 - 4$