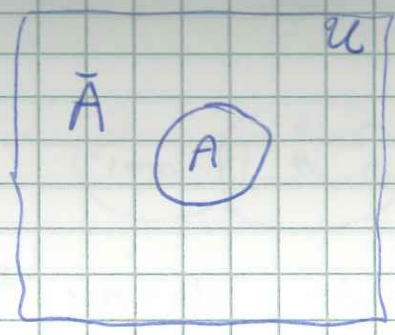
Chapter 8 - Inclusion - Exclusion

Venn Diagrams If A is a set, $\bar{A}$ is all things not in A

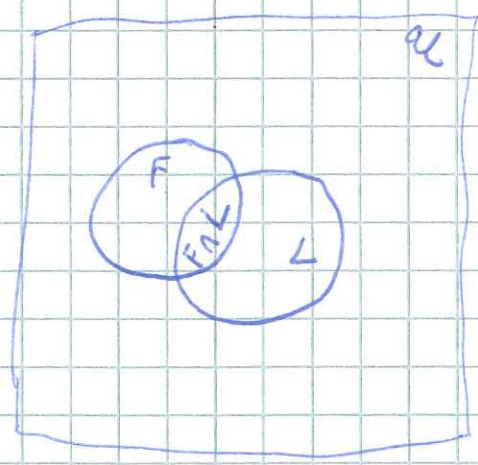
often written A^c in the literature



$N(A) = |A| = \# A =$ # of elements in A

$N = N(\mathcal{U}) =$ total # of elements in a universe $\mathcal{U}$

$N(\bar{A}) = N - N(A)$



$$N(F \cup L) = N(F) + N(L) - N(F \cap L)$$

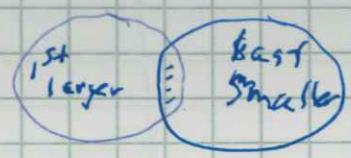
De Morgan's Law: $\bar{F} \cap \bar{L} = \overline{F \cup L}$

so $N(\bar{F} \cap \bar{L}) = N(\overline{F \cup L}) = N - N(F \cup L)$

so $N(\bar{F} \cap \bar{L}) = N - (N(F) + N(L) - N(F \cap L)) = N - N(F) - N(L) + N(F \cap L)$

ex. How many arrangements of the digits 0, 1, 2, ..., 9 are there in which the 1st digit is larger than one

(ten digit #'s, all digits different)



F = set $1^{st} > 1$

L = set last < 8

and the last digit smaller than 8?

$$\underbrace{\#}_{N(F\cap L)} = \underbrace{(\# \ 1^{st} > 1)}_{N(F)} + \underbrace{(\# \ \text{last} < 8)}_{N(L)} - \underbrace{(\# \ 1^{st} > 1 \text{ or last} < 8)}_{N(F\cup L)}$$

$$= 8 \cdot 9! + 8 \cdot 9! - (\# \ 1^{st} > 1 \text{ or last} < 8)$$

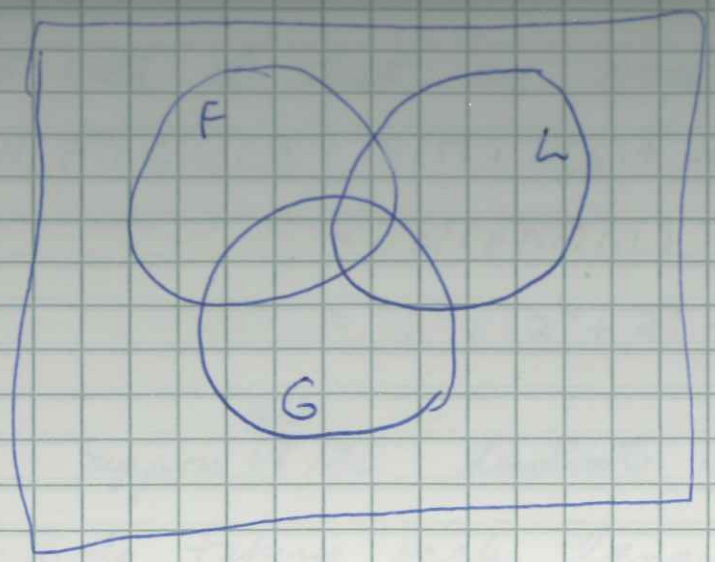
$$\underbrace{\# \ 1^{st} > 1 \text{ or last} < 8}_{N(F\cup L)} = \underbrace{\text{total} \ \#}_{N} - \underbrace{(\# \ 1^{st} \leq 1 \text{ or last} \geq 8)}_{N(\overline{F\cup L})}$$

$$= \text{total} \ \# - [\# \ 1^{st} \leq 1 \text{ and } \# \ \text{last} \geq 8]$$

$$= 10! - [\qquad 2 \cdot 2 \cdot 8!]$$

Final Answer: $2 \cdot 8 \cdot 9! - [10! \qquad - 4 \cdot 8!] = 8!(16 \cdot 9 - 90 + 4)$

$= 8!(58)$



$$N(F \cup L \cup G) = N(F) + N(G) + N(L) - N(F \cap L) - N(F \cap G) - N(L \cap G) + N(F \cap L \cap G)$$

$$N(\bar{F} \cap \bar{L} \cap \bar{G}) = N - N(F) - N(L) - N(G) + N(F \cap L) + N(F \cap G) + N(L \cap G) - N(F \cap L \cap G)$$

ex. How many n-digit ternary (0, 1, 2) sequences are there with at least one 0, at least one 1, and at least one 2? How many n-digit ternary sequences with at least one void (missing digit)?

F = set with no 0's, G = set with no 1's,

L = set with no 2's

$$N(\bar{F} \cap \bar{L} \cap \bar{G}) = N - N(F) - N(L) - N(G) + N(F \cap L) + N(F \cap G) + N(L \cap G)$$

$$= 3^n - 2^n - 2^n - 2^n + 1 + 1 + 1 = \boxed{3^n - 3 \cdot 2^n + 3}$$

with at least one void

$$N(F \cup G \cup L) = N(F) + N(G) + N(L) - N(F \cap G) - N(F \cap L) - N(G \cap L) + N(F \cap G \cap L)$$

$$= 2^n + 2^n + 2^n - 1 - 1 - 1 + 0 = \boxed{3 \cdot 2^n - 3}$$

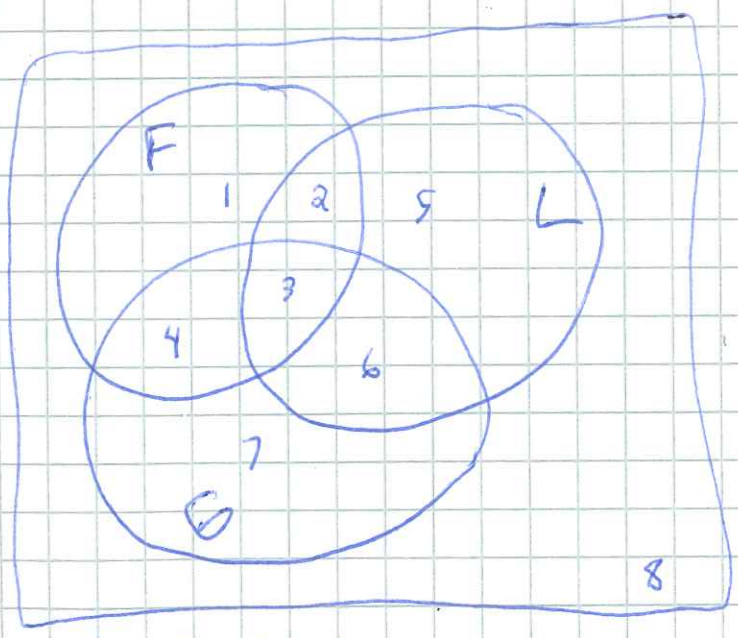
ex Suppose there are 100 students in a school + 40 students taking each language French, Latin, and German. 20 students are taking only French, 20 only Latin, 15 only German. 10 are taking French and Latin. How many are taking all 3? No language?

F = French Students, G = German Students,

L = Latin Students

$N = 100$

$N(F) = N(L) = N(G) = 40$



$N(F \cap \bar{L} \cap \bar{G}) = 20 = N_1$

$N(L \cap \bar{F} \cap \bar{G}) = 20 = N_5$

$N(G \cap \bar{F} \cap \bar{L}) = 15 = N_7$

$N(F \cap L) = 10 = N_2 + N_3$

$N_1 + N_2 + N_3 + N_4 = 40$ $\quad N_2 + N_3 = 10$

$N_1 = 20 \Rightarrow \boxed{N_4 = 40 - 20 - 10 = 10}$

$N_5 = 20 \quad N_7 = 15$

$N_2 + N_3 + N_5 + N_6 = 40$

$N_5 = 20 \quad N_2 + N_3 = 10 \Rightarrow \boxed{N_6 = 10}$

$N_3 + N_4 + N_6 + N_7 = 40$

$N_7 = 15 \quad N_4 + N_6 = 20 \Rightarrow \boxed{N_3 = 5}$

So 5 students are taking all 3.

$N_2 + N_3 = 10$, $N_3 = 5 \Rightarrow N_2 = 5$

$\Rightarrow N - (N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7)$

= # taking no language

$= 100 - (20 + 5 + 5 + 10 + 20 + 10 + 15)$

$= 100 - 85 = \boxed{15} = N_8$

ex. #3. How many n-digit ternary sequences are there in which at least one pair of consecutive digits are the same?

# of there = total # − (# no consecutive digits are the same)

$$= 3^n - \underset{\text{choice digit 1}}{3} \cdot \underset{\text{choice digit 2}}{2} \cdot \underset{\text{choice digit 3}}{2} \cdot \quad \cdot \underset{\text{choice digit n}}{2}$$

$$= \boxed{3^n - 3 \cdot 2^{n-1}}$$

#7. If you pick a random integer between 1 & 1,000, what is the probability that it is either divisible by 7 or even (or both)? $571/1000$

A = set divisible by 7 B = set even

$N(A \cup B) = N(A) + N(B) - N(A \cap B)$

$= [1000/7] + [1000/2] - [1000/14] = 142 + 500 - 71 = 571$

ex #27. How many arrangements are there of TAMELY with either T before A, or A before M, or M before E? (by before we mean anywhere before

F = set with T before A

G = " " A before M

H = set with M before E

$$N(F \cup G \cup H) = N(F) + N(G) + N(H) - N(F \cap G) - N(F \cap H) - N(G \cap H) + N(F \cap G \cap H)$$

$$3 \cdot \binom{6}{2} 4! - \binom{6}{3} 3! - \binom{6}{2}\binom{4}{2} 2! - \binom{6}{3} 3! + \binom{6}{4} 2!$$

choose T + A letters

choose A + M "

ex. #18. How many numbers between 1 and 280 are relatively prime to 280?

$280 = 4 \cdot 70 = 8 \cdot 5 \cdot 7$

rel. prime = total # − (multiples of 2) − (multiples of 5) − (multiples of 7) + (multiples of 10) + (multiples of 14) + (multiples of 35) − (multiples of 70) = 280 − 140 − 56 − 40 + 28 + 20 + 8 − 4

$= 280(1 - \frac{1}{2})(1 - \frac{1}{5})(1 - \frac{1}{7}) = 280 \cdot \frac{1}{2} \cdot \frac{4}{5} \cdot \frac{6}{7} = 4 \cdot 24 = 96$

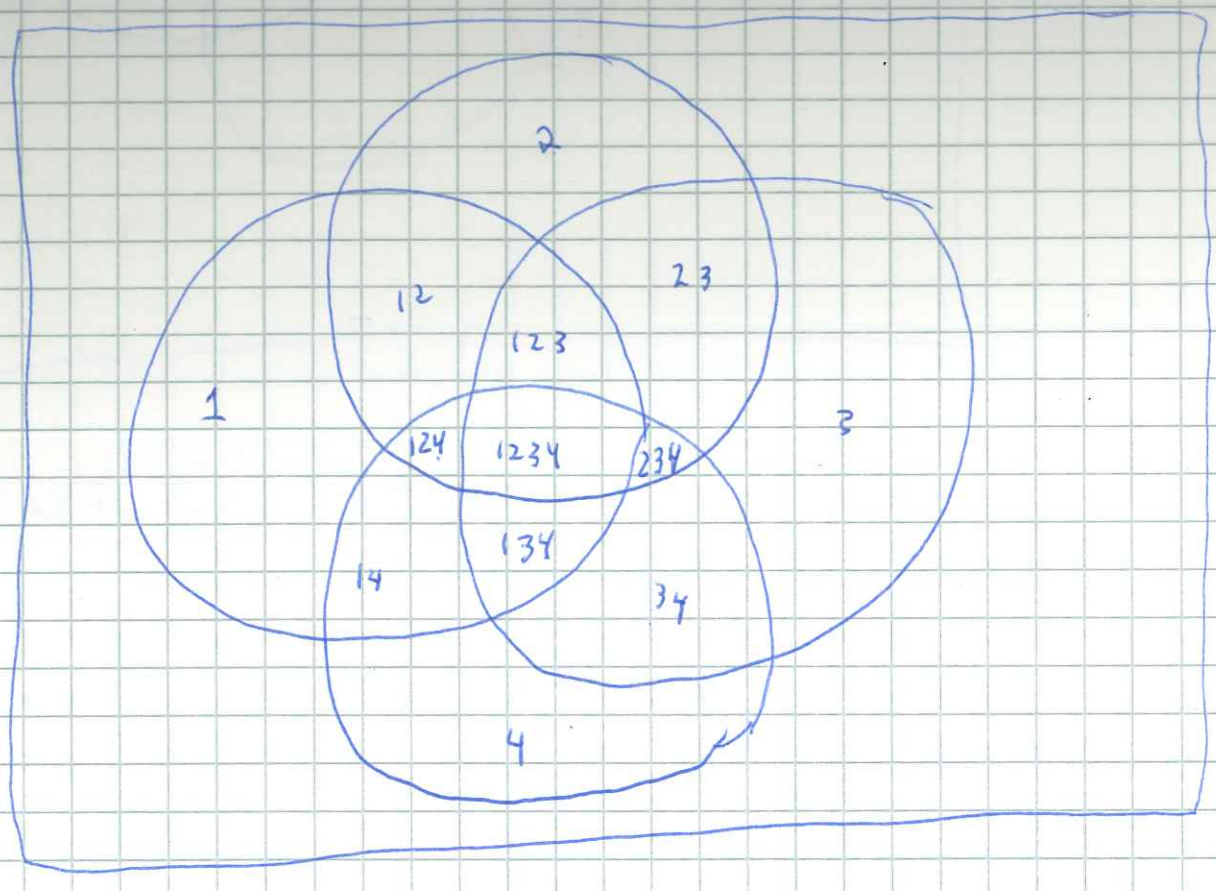
More generally, $\phi(n)$ = # relatively prime to n and between 1 and n $= n \prod_{p|n} (1 - 1/p)$ (Euler's totient function)

8.7

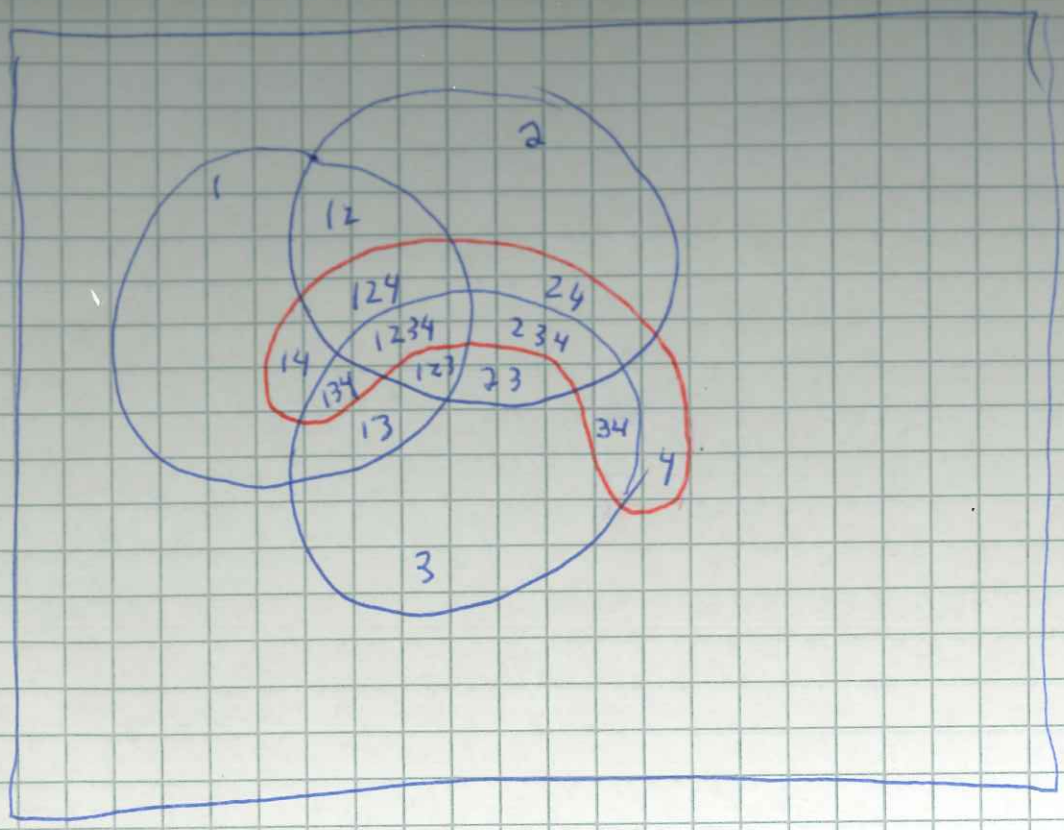
$$N(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n) = N - S_1 + S_2 - S_3 \cdots + (-1)^n S_n$$

where S_k is the sum of $N(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$, over all $1 \le i_1 < i_2 < \cdots < i_k \le n$

ex. $n=4$



you can't get all possible elements of each S_k ... using circles, but you can using more general shapes



Also,

$$N(A_1 \cup A_2 \cup \cdots \cup A_n) = S_1 - S_2 + S_3 - \cdots + (-1)^{n-1} S_n$$

ex. How many ways to select a 6-card hand from a regular 52-card deck such that the hand contains at least one card in each suit?

Let S = set of hands with at least one spade
D = " " " " diamond
H = " " " " heart
C = " " " " club

$$N(S \cap D \cap H \cap C) = N - S_1 + S_2 - S_3 + S_4 = \boxed{\binom{52}{6} - 4\cdot\binom{39}{6} + 6\cdot\binom{26}{6} - 4\cdot\binom{13}{6}}$$

$S_1 = N(S^c) + N(D^c) + N(H^c) + N(C^c) = 4\cdot\binom{39}{6}$ $\qquad S_4 = 0$

$S_2 = N(S^c \cap D^c) + \cdots + N(H^c \cap C^c) = 6\cdot\binom{26}{6}$ $\qquad N = \binom{52}{6}$

$S_3 = N(S^c \cap D^c \cap H^c) + \cdots + N(D^c \cap H^c \cap C^c) = 4\cdot\binom{13}{6}$

8.2 (continued)

ex. How many integer sol'ns to

$x_1 + x_2 + \cdots + x_6 = 20 \qquad 0 \le x_i \le 8$

gen. func. sol'n $(1+x+x^2+\cdots+x^8)^6 \Big|_{x^{20}} = \left(\frac{1-x^9}{1-x}\right)^6 \Big|_{x^{20}}$

Let A_i = # sol'ns to $x_1 + \cdots + x_6 = 20 \qquad x_i \ge 9$

other $x_j \ge 0$

We want $N(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_6)$

Now $N(\bar{A}_1 \cap \cdots \cap \bar{A}_6) = N - S_1 + S_2 - S_3 + S_4 - S_5 + S_6$

Note that if at least three of the x_i are ≥ 9 then the sum is ≥ 27, so $S_3 = S_4 = S_5 = S_6 = 0$

$N = \binom{20+5}{5} \qquad S_1 = \binom{6}{1}\binom{11+5}{5} \qquad S_2 = \binom{6}{2}\binom{2+5}{5}$

so answer is $\boxed{\binom{25}{5} - 6 \cdot \binom{16}{5} + 15 \cdot \binom{7}{5}}$

ex. What is the probability that if n people randomly reach into a dark closet to retrieve their hats, no person will retrieve their own hat?

A <u>derangement</u> of the numbers 1 thru n is a permutation σ of 1 thru n where $\sigma_i \ne i$ for $1 \le i \le n$

What is the probability that a random permutation of $1, 2, \ldots, n$ is a derangement (i.e. has no fixed pts)?

Let A_i = set of σ with $\sigma_i = i$, so $N(A_i) = (n-1)!$

$N(A_i \cap A_j) = (n-2)!$ $\quad$ $N(A_i \cap A_j \cap A_k) = (n-3)!$, etc.

$1 \le i < j \le n$ $\quad$ $1 \le i < j < k \le n$

$$N(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n) = N - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k + \cdots + (-1)^n S_n$$

$$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^k \binom{n}{k}(n-k)! + \cdots + (-1)^n$$

$$= n! - \frac{n}{1!}(n-1)! + \frac{n(n-1)}{2!}(n-2)! + \cdots + (-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!}(n-k)! + \cdots + (-1)^n$$

$$= n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!} + \cdots + \frac{(-1)^n}{n!}\right]$$

so probability is $\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \cdots \frac{(-1)^n}{n!} \cong \frac{1}{e}$

(Taylors series for $\frac{1}{e}$)

Ex. 3 What is the prob. that a 10-card hand from a standard 52-card deck has at least one 4 of a kind?

Let A_i = set of 10 card hands with four cards of face value i (aces face value 1, ..., kings face value 13)

$1 \le i \le 13$. Then $S_1 - S_2 = \binom{13}{1}\binom{48}{6} - \binom{13}{2}\binom{44}{2}$

$$\text{prob.} = \frac{\binom{13}{1}\binom{48}{6} - \binom{13}{2}\binom{44}{2}}{\binom{52}{10}}$$

ex. #13 How many ways for a child to take 10 pieces of candy, if the child does not take exactly 7 piece of candy of any type, where there are 4 types of candy?

Let A_i = set of ways to take 10 pieces of candy with exactly 7 pieces of type i $\quad 1 \le i \le 4$

Clearly $N(A_i)$ = # sol'n's to $x_1 + x_2 + x_3 + x_4 = 10$, $x_i = 7$, other $x_j \ge 0$, so $N(A_1)$ = # sol'n's to $x_2 + x_3 + x_4 = 8$, $x_2, x_3, x_4 \ge 0$

$= \binom{8+2}{2}$

$N(A_1 \cap A_2)$ = # sol'n's to $x_3 + x_4 = 6$, $x_3, x_4 \ge 0$

$= \binom{6+1}{1}$

$N(A_1 \cap A_2 \cap A_3) = 1$

N = # sol'n's to $x_1 + x_2 + x_3 + x_4 = 10$, all $x_i \ge 0$

$= \binom{10+3}{3}$

$S_1 = \binom{4}{1}\binom{10}{2}$

$S_2 = \binom{4}{2} 7$

$S_3 = \binom{4}{3} \cdot 1$

$S_4 = 0$

$N(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4) = N - S_1 + S_2 - S_3 + S_4$

$= \binom{13}{3} - 4 \cdot \binom{10}{2} + 6 \cdot 7 - 4$