A board $B$ is a subset of an $n \times n$ grid of squares. We can think of the squares of $B$ as "forbidden positions" in some permutation

$$B = \{(1,1), (2,2), (3,3), (4,4)\}$$

How many permutations in $S_n$ avoid the squares of $B$?

This is the Derangement problem for $n = 4$.

A rook placement is a selection of the squares in the $n \times n$ grid with no two squares in the same row or column. A permutation $\sigma \in S_n$ which avoids the forbidden positions encoded by the squares of $B$ can be thought of as a placement of $n$ rooks none of them on $B$. (In chess, rooks attack everything in their row or column). More generally, let $t_k(B)$, the $k$th hit number of $B$, be the # of placements of $n$ rooks on the $n \times n$ grid with exactly $k$ rooks on the squares of $B$, i.e., the # of $\sigma \in S_n$ which violate $k$ of the forbidden positions.
Ex. \( B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \) 
\( t_0 = 3 \) 
\( t_1 = 2 \) 
\( t_2 = 1 \)

Rook Theory expresses the hit numbers in terms of the rook numbers \( r_k(B) \), where \( r_k(B) = \# \text{ of placements of } k \text{ rooks on the squares of } B \), where \( r_0(B) = 1 \) for all \( B \)

Ex. \( B = \begin{bmatrix} \text{ } & \text{ } & \text{ } \\ \text{ } & \text{ } & \text{ } \end{bmatrix} \) 
\( r_1(B) = 4 = \# \text{ of squares in } B \) 
\( r_2(B) = 1 \) 
\( r_3(B) = 1 \) 
\( r_4(B) = 0 \)
Then (Riordan-Kaplanovski) For any matrix $B$,

$$
\sum_{k=0}^{n} t_k(B) (x-1)^{n-k} \frac{1}{n-k} = \sum_{i=0}^{n} t_i(B) x^i
$$

Proof. For an $n \times n$ matrix $A$, the permanent of $A$, denoted $\text{per}(A)$, is defined as

$$
\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i, \sigma(i)}
$$

Note that

$$
\det(A) = \sum_{\sigma \in S_n} \left( \prod_{i=1}^{n} a_{i, \sigma(i)} \right) \text{sgn}(\sigma)
$$

so $\text{per}(A) = \det(A)$ without the minus signs.

Example: $\text{per}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc$

$$
\text{per}\begin{pmatrix} a & b & c \\ d & e & f \\ u & v & w \end{pmatrix} = aew + afu + bdw + bfu + cdv + cew
$$

The permanent of $A$ is important in theoretical computer science and also in algebra and combinatorics.

The hit polynomial of $B$, $\sum_{i=0}^{\infty} t_i(B) x^i$, can be expressed as the permanent of the matrix $A$ where

$$
(A)_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in B \\ 0 & \text{otherwise} \end{cases}
$$
\[ \text{ex} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

hit poly = \(3 + 2x + x^2\)

\[
\text{per} \begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix}
\]

To prove \( \Theta \), expand this permanent in powers of \( x-1 \):

\[
\text{per} \begin{bmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \text{per} \begin{bmatrix} x-1+1 & 1 \\ 1 & x-1+1 \\ 1 & 1 \end{bmatrix}
\]

coeff of \((x-1)^k\) is \( r_k(B)(n-k)! \)

Can \( t_{r\theta}(B) = \# \sigma \in S_n \text{ avoiding all forbidden positions} \)

\[
= \sum_{k=0}^{n} r_k(B)(-1)^k (n-k)!
\]

If set \( k = 0 \) in \( \otimes \)

It is typically easier to compute the \( r_k(B) \) than the \( t_{r\theta}(B) \)
Let \( B = \{ (i, j) \mid 1 \leq i \leq n \} \)

Then \( r_k(B) = \binom{n}{k} \), and \( \text{to}(B) = \# \text{of derangements of } n \text{ things} \leq D_n \)

So \( D_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! \)

\[ = n! \left( \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right) \]

Calculating the rook numbers

If \( B = S \cup T \), where no squares in \( S \) share a row or column with any squares in \( T \), then

\[ r_k(S \cup T) = \sum_{j=0}^{k} \binom{k}{j} r_j(S) r_{k-j}(T) \quad 0 \leq k \leq n \]

\[ S = \{ (1,1), (1,2), (3,2), (4,2) \} \]

\[ T = \{ (3,4), (3,5), (5,4) \} \]

\[ r_2(B) = r_2(5) r_0(T) + r_1(5) r_1(T) + r_0(3) r_2(S) \]

\[ = 2 \cdot 1 + 4 \cdot 3 + 1 \cdot 1 = 15 \]

The rook polynomial of \( B \) is defined as

\[ R(B) = \sum_{k=0}^{n} r_k(B) x^k \]

Identity \( \ast \ast \) is equivalent to

\[ R(S \cup T) = R(5) R(T) \]
ex. \( B = \begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
& & & \\
\hline
\end{array} \)

\( S = \{(1,1), (1,2), (2,2), (4,2)\} \)

\( T = \{(3,4), (3,5), (5,4)\} \)

\[
R(s) = 1 + 4x + 2x^2
\]

\[
R(T) = 1 + 3x + x^2
\]

\[
R(B) = (1 + 4x + 2x^2)(1 + 3x + x^2) = 1 + 7x + 15x^2 + 10x^3 + 2x^4
\]

Another useful trick: Let \( s \in B \), and let \( B_s \) be the board \( B \) with all squares from \( B \) in the column on row containing \( s \) removed.

Then \( R_k(B) = R_{k-1}(B_s) + R_k(B_{1s}) \)

where \( B_{1s} \) is \( B \) with \( s \) removed.

ex. for \( B = \begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
& & & \\
\hline
\end{array} \)

\( S = \{(2,2)\} \)

\( B_s = \begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
& & & \\
\hline
\end{array} \)

\( B_{1s} = \begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
& & & \\
\hline
\end{array} \)

\[
\Gamma(B) = \Gamma(B_s) + \Gamma(B_{1s})
\]

\( \Gamma(\text{no rook on } s) = \Gamma(\text{rook on } s) \)
Calculate the number of words that can be formed by rearranging the letters E, E, E, R, I, E so that no letter occupies one of its original positions. For example, no E in the 1st, second, or 6th letters.

\[ R(6) = R(2) \cdot R(4) \cdot R(2) \]

\[ R(6) = \left(1 + 9x + 33x^2 + 6x^3\right) \left(1 + 4x + 12x^2\right) (1 + x) \]

We want to divide by 6.

\[ R(2) = (1 + 9x + 18x^2 + 6x^3) (1 + 5x + 6x^2 + 2x^3) \]

\[ R(4) = (1 + 14x + 69x^2 + 152x^3 + 156x^4 + 72x^5 + 12x^6) \]

\[ t_0(6) = 1 \]

\[ t_0(2) = 1 \]

\[ t_0(4) = 1 \]

\[ \frac{t_0(6)}{12} = 3 \]

**Ex. Probleme des Ménages**

How many ways can we seat n married man-woman couples around a circular table with 2n chairs, alternating man-woman, so that no husband sits next to his wife? Let \( M_n \) be the number of ways.

Begin by seating the women in \( 2n \) ways. Label the men \( M_1, M_2, \ldots, M_n \), where

\[ M_n \] is married to woman \( i \).
Labelling the chairs \(1, 2, \ldots, n, n\) as we go around the table, seat women in seat \(i\) for \(1 \leq i \leq n\).

Then

\[ M_n = 2 \cdot n! \cdot \alpha_n \quad \text{where} \]

\[ \alpha_n = \text{# ways to seat men in chairs } 1, 2, \ldots, n \]

with men not in chairs 1 or 2, men not in chairs 2 or 3, \ldots, men not in chairs \(n-1\) or \(n\), and man \(n\) not in chairs \(n\) or 1.

Now \( \alpha_n = \beta_n \cdot \beta_{n-1} \ldots \beta_1 \) where \( \beta_{i+1} \) is the board.
Note if $s$ is the square $(n,1)$, then

$$r_k(B_s^{(n)}) = r_{k+1}(B_{s/5}^{(n)}) + r_k(B_{s/5}^{(n)})$$

by $\star$.

Now $r_{k-1}(B_{s/5}^{(n)})$ is the number of ways to choose $k-1$ dots from a sequence of $n+3$ dots, no two consecutive, and $r_k(B_{s/5}^{(n)})$ is the number of ways to choose $k$ dots, no two consecutive, from a sequence of $n+1$ dots.

**Lemma** The # of ways to choose $p$ dots from a sequence of $m$ dots, no two consecutive, is

$$\binom{m-p+1}{p}$$

**PS.** Begin by placing $p$ dots in a line

$$\ldots x_1 x_2 x_3 x_4 \ldots x_p x_{p+1}$$

Place remaining dots in boxes, between dots or before or after the dots.
\[ x_1 + x_2 + \cdots + x_{p+1} = m - p \quad x_i, x_{p+1} \geq 0 \quad x_2, x_3, \ldots, x_p \geq 1 \]

The number of solutions is:

\[ \binom{m-p+1}{p} = \binom{m-p+1}{p} \]

Thus,

\[ r_k(B_{2n}) = r_k(B_{s}^{(m)}) + r_k(B_{s}^{(n)}) \]

\[ = \binom{3n-3-(k-1)+1}{k-1} + \binom{2n-k+1}{k} \]

\[ = \binom{3n-k-1}{k-1} + \binom{2n-k}{k} = \frac{(2n-k-1)!}{(k-1)! (2n-k)!} + \frac{(2n-k)!}{k! (2n-2k)!} \]

\[ = \frac{(2n-k)!}{k! (2n-2k)!} \left[ K \frac{K}{2n-k} + 1 \right] = \binom{3n-k}{k} \frac{2n}{2n-k} \frac{n-k}{(n-k)!} (-1)^k \text{ by } \square \]

Thus,

\[ M_n = 2n! \sum_{k=0}^{n} \binom{2n-k}{k} \frac{2n}{2n-k} \frac{n-k}{(n-k)!} (-1)^k \]

**Ex. #8.** Suppose five officials O₁, O₂, O₃, O₄, O₅ are to be assigned five different cars: an Escort, a Lexus, a Nissan, a Taurus, and a Volvo. (E, L, N, T, V)

O₁ will not drive an Escort or a Taurus.

O₂ will drive a Taurus.

O₃ will drive a Lexus or a Volvo.

O₄ will drive a Lexus.

O₅ will drive an Escort or a Nissan.
If a feasible assignment of cars is chosen randomly, what is the probability that
(a) 0₁ gets the Volvo?
(b) 0₂ or 0₃ get the Volvo?

\[ S = \{ (0₁, E), (0₁, T), (0₂, E), (0₂, N), (0₂, T) \} \]
\[ T = \{ (0₃, L), (0₃, V), (0₄, L) \} \]

(c) If 0₁ gets the Volvo, then the remaining board is

\[ R(B) = R(5) \cdot P(T) = (1 + 5x + 6x^2 + x^3)(1 + 3x + x^2) \]
\[ = 1 + 8x + 32x^2 + 24x^3 + 9x^4 + x^5 \]
\[ t₀(B) = 5!, - 8 \cdot 4! + 22 \cdot 3! - 24 \cdot 2! + 9 \cdot 1! - 0! = 20 \]
\[ R(B') = 1 + 5x + 8x^2 + 4x^3 \]
\[ t₀(B') = 4!, - 5 \cdot 3! + 8 \cdot 2! - 4 \cdot 1! = 6 \]

\[ \text{Prob.} = \frac{t₀(B')}{t₀(B)} = \frac{6}{20} = 30\% \]
(b) If $O_2$ gets the Volvo, remaining board is

\[ S = \{(O_1, E), (O_1, T), (O_3, E), (O_3, N)\} \]

\[ T = \{(O_2, L), (O_4, L)\} \]

\[ R(S) = 1 + 4x + 3x^2 \quad R(T) = 1 + 3x \]

\[ R(B''') = (1 + 4x + 3x^2)(1 + 3x) = 1 + 6x + 11x^2 + 6x^3 \]

\[ t_0(B''') = 8 - 6.3! + 1.2! - 6.1! = 24 - 76 + 72 - 6 = 4 \]

Probability $O_2$ gets $V$ is \[ \frac{4}{30} = 0.09 \]

If $O_3$ gets the Volvo, remaining board is

\[ R(B''') = (1 + 3x + x^2)(1 + 2x) \]

\[ = 1 + 5x + 7x^2 + 2x^3 \]

\[ t_0(B''') = 24 - 5.6 + 7.7 - 7.1 = 24.30 + 14 - 2 = 6 \]

Probability $O_2$ or $O_3$ get Volvo is \[ \frac{4 + 6}{30} = 0.50 \]
