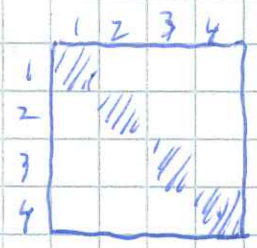
8.3 Rook Polynomials

A board B is a subset of an $n \times n$ grid of squares. We can think of the squares of B as "forbidden positions" in some permutation

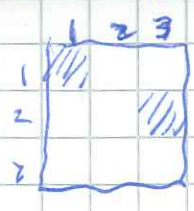
ex: $B = \{(1,1), (2,2), (3,3), (4,4)\}$



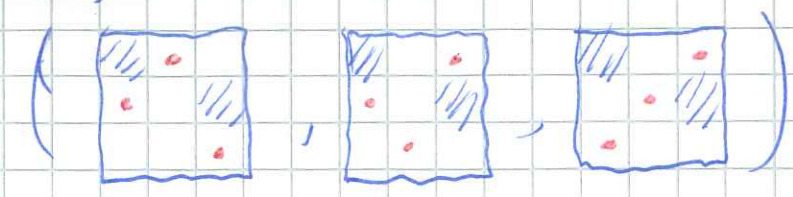
How many permutations in S_4 avoid the squares of B?
This is the Derangement problem for $n=4$

A <u>rook placement</u> is a selection of the squares in the $n \times n$ grid with no two squares in the same row or column. A permutation $\sigma \in S_n$ which avoids the forbidden positions encoded by the squares of B can be thought of as a placement of n rooks, none of them on B. (in chess, rooks attack everything in their row or column). More generally, let $t_k(B)$, the kth hit number of B, be the # of placements of n rooks on the $n \times n$ grid with exactly k rooks on the squares of B, i.e. the # of $\sigma \in S_n$ which violate k of the forbidden positions

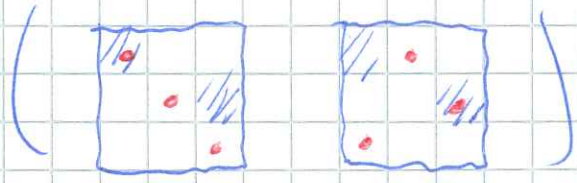
ex. $B = \{(1,1), (2,3)\}$



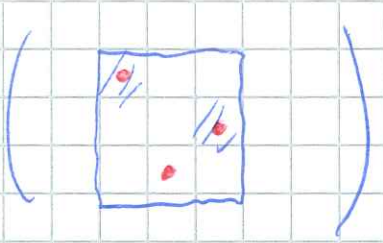
$t_0 = 3$



$t_1 = 2$



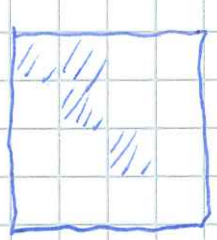
$t_2 = 1$



Rook Theory expresses the hit numbers in terms of the rook numbers $r_k(B)$, where

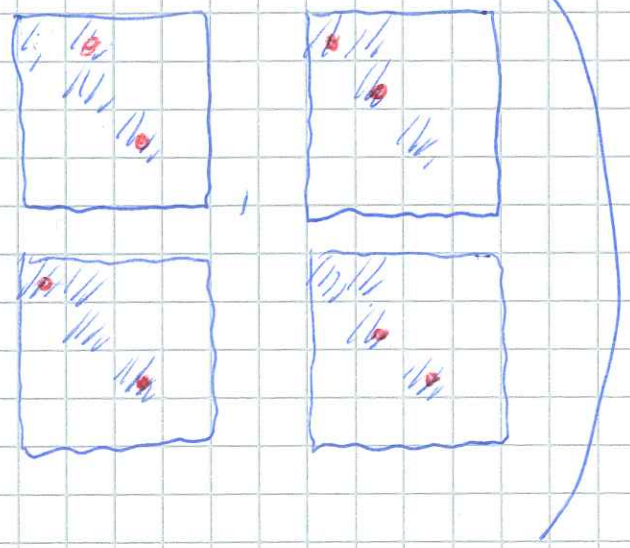
$r_k(B)$ = # of placements of k rooks on the squares of B, where $r_0(B) = 1$ for all B

ex. $B =$

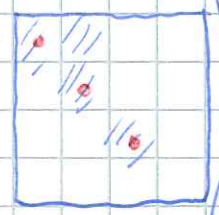


$r_1(B) = 4$ = # of squares in B

$r_2(B) = 4$



$r_3(B) = 1$



$r_4(B) = 0$

Thm (Riordan - Kaplansky) For any board B,

$$\sum_{k=0}^{n} r_k(B)\,(x-1)^k\,(n-k)! = \sum_{j=0}^{n} t_j(B)\,x^j \qquad (*)$$

<u>Proof.</u> For an $n \times n$ matrix A, the permanent of A, denoted $\mathrm{per}(A)$, is defined as

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma_i}$$

Note that $\det(A) = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i,\sigma_i} \right) (-1)^{\mathrm{sgn}(\sigma)}$

so $\mathrm{per}(A) = \det(A)$ without the minus signs

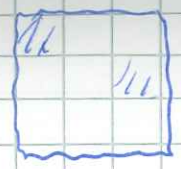
ex. $\mathrm{per}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad + bc$

$$\mathrm{per}\begin{bmatrix} a & b & c \\ d & e & f \\ u & v & w \end{bmatrix} = aew + afv + bdw + bfu + cdv + ceu$$

per A is important in theoretical computer science and also in algebra and combinatorics

The <u>hit</u> <u>polynomial</u> of B, $\sum_{j=0}^{n} t_j(B)\,x^j$, can be expressed as the permanent of the matrix A where

$$(A)_{ij} = \begin{cases} 1 & \text{if } (i,j) \notin B \\ x & \text{if } (i,j) \in B \end{cases}$$

ex. $B =$  hit poly $= 3 + 2x + x^2$

$$= \operatorname{per}\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix}$$

To prove ⊛, Expand this permanent in powers of $x-1$:

$$\operatorname{per}\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix} = \operatorname{per}\begin{bmatrix} x-1+1 & 1 & 1 \\ 1 & 1 & x-1+1 \\ 1 & 1 & 1 \end{bmatrix}$$

coef of $(x-1)^k$ is $r_k(B)\,(n-k)!$ ∎

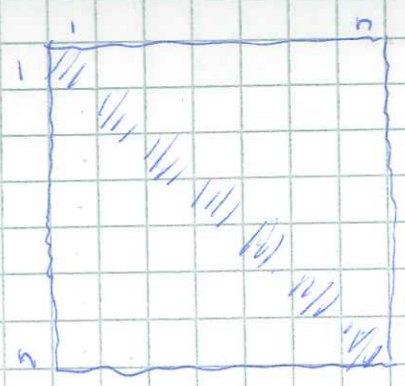
<u>Cor</u> $t_0(B) = \#\,\sigma \in S_n$ avoiding all forbidden positions

$$= \sum_{k=0}^{n} r_k(B)\,(-1)^k\,(n-k)!$$

pf set $x=0$ in ⊛ ∎

It is typically easier to compute the $r_k(B)$ than the $t_j(B)$

ex. Let $B = \{(i,i) \quad 1 \le i \le n\}$



Then $r_k(B) = \binom{n}{k}$, and $t_0(B) = \#$ of derangements of n things $= D_n$

So $D_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)!$

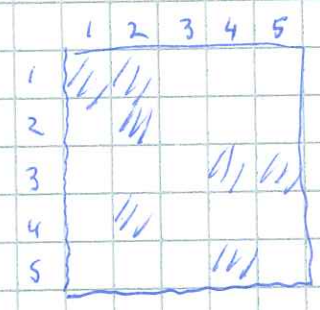
$= n! \left[\sum_{k=0}^{n} \frac{(-1)^k}{k!} \right]$

<u>Calculating the rook numbers</u>

If $B = S \cup T$, where no squares in S share a row or column with any squares in T, then

$$r_k(S \cup T) = \sum_{j=0}^{k} r_j(S)\, r_{k-j}(T) \qquad 0 \le k \le n \qquad (**)$$

ex.



$S = \{(1,1), (1,2), (2,2), (4,2)\}$

$T = \{(3,4), (3,5), (5,4)\}$

$r_2(B) = r_2(S)\, r_0(T) + r_1(S)\, r_1(T) + r_0(S)\, r_2(T)$

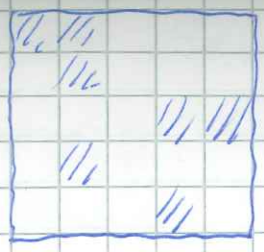
$= 2 \cdot 1 + 4 \cdot 3 + 1 \cdot 1 = 15$

The <u>rook</u> polynomial of B is defined as

$$R(B) = \sum_{k=0}^{n} r_k(B)\, x^k$$

Identity $(**)$ is equivalent to

$$R(S \cup T) = R(S)\, R(T)$$

ex. $B =$ 

$S = \{(1,1), (1,2), (2,2), (4,2)\}$

$T = \{(3,4), (3,5), (5,4)\}$

$R(S) = 1 + 4x + 2x^2$

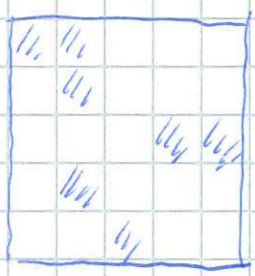
$R(T) = 1 + 3x + x^2$

$R(B) = (1+4x+2x^2)(1+3x+x^2) = 1 + 7x + 15x^2 + 10x^3 + 2x^4$

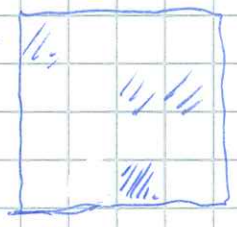
Another useful trick: Let $s \in B$, and let B_s be the board B with all squares from B in the column or row containing s removed.

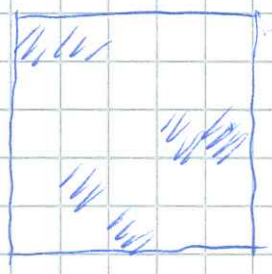
Then $r_k(B) = r_{k-1}(B_s) + r_k(B_{/s})$, (***)

where $B_{/s}$ is B with s removed.

ex. for $B =$ 

$S = (2,2)$

$B_s =$ 

$B_{/s} =$ 

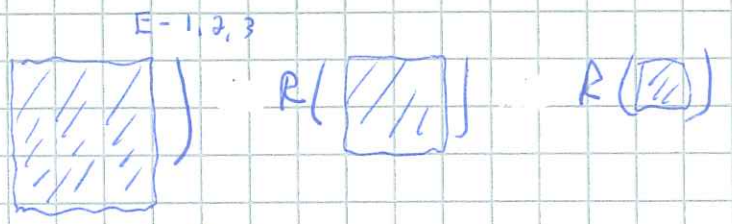
$r_2(B) = r_1(B_s) + r_2(B_{/s})$

rook on s

no rook on s

#9

ex. Calculate the # of words that can be formed by rearranging the letters EERRIE so that no letter occupies one of its original positions — for example, no E in the 1st, second, or 6th letter



$= (1+9x+3\cdot 3\cdot 2x^2+6x^3)(1+4x+2x^2)(1+x)$

we want $t_0(B)/3!\,2!$

$18+45+6$

$6\ 90\ 54\ 2$

$30\ 108$

36

$R(B) = (1+9x+18x^2+6x^3)(1+5x+6x^2+2x^3)$

$R(B) = 1+14x+69x^2+152x^3+156x^4+72x^5+12x^6$

$t_0(B) = 6! - 14\cdot 5! + 69\cdot 4! - 152\cdot 3! + 156\cdot 2! - 72 + 12 = 36$

$\frac{t_0(B)}{12} = 3$

ex. Problème des Ménages

How many ways can we seat n married man-woman couples around a circular table with $2n$ chairs, alternating man-woman, so that no husband sits next to his wife? Let M_n be the number of ways

Begin by seating the women in $2\cdot n!$ ways.

Label the men & women as $m_1, m_2, \ldots m_n$ and $w_1, w_2, \ldots w_n$, where

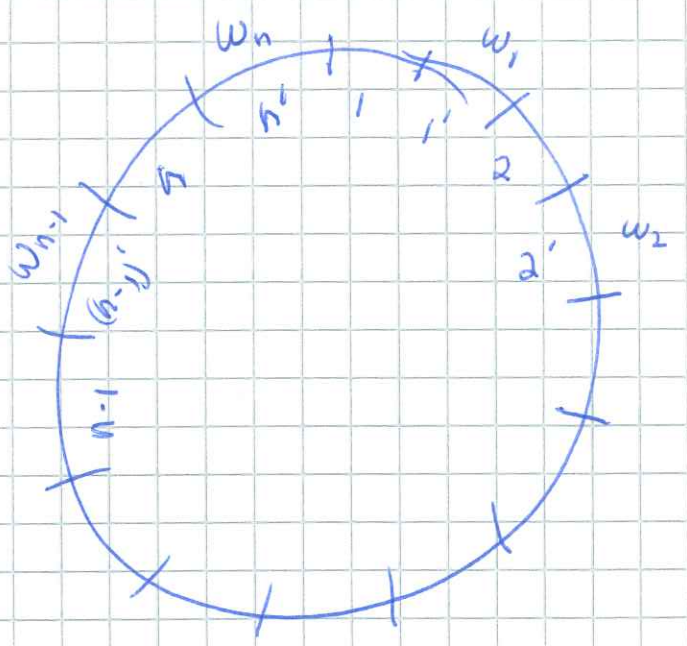
man i is married to woman i $\quad 1 \le i \le n$

Labelling the chairs $1, 1', \ldots, n, n'$ as we go around the table, seat woman$_i$ in seat i' for $1 \le i \le n$

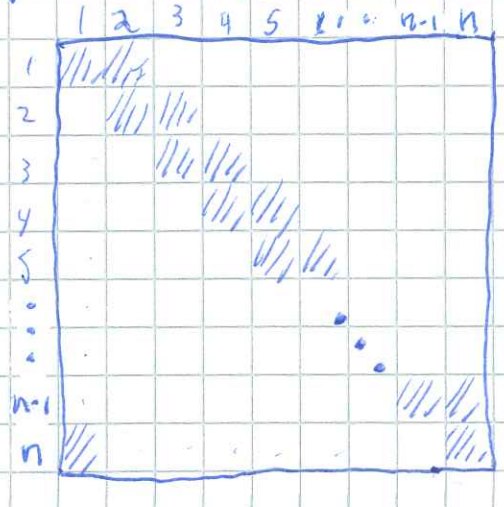
Then

$M_n = 2 \cdot n! \, \alpha_n$, where

α_n = # ways to seat men in chairs $1, 2, \ldots, n$ with man$_1$ not in chairs 1 or 2, man$_2$ not in chairs 2 or 3, $\ldots$, man$_{n-1}$ not in chairs $n-1$ or n, and man$_n$ not in chairs n or 1.

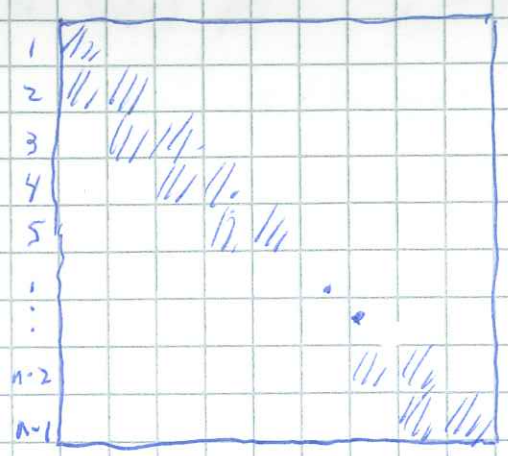


Now $\alpha_n = t_0(B^{(n)})$, where $B^{(n)}$ is the board

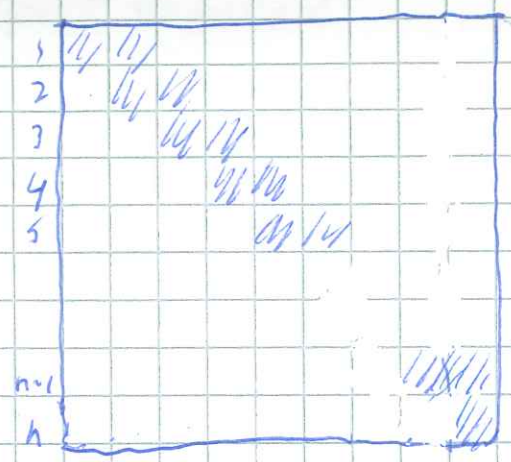


Note if s is the square $(n,1)$, then

$$r_k(B^{(n)}) = r_{k-1}(B^{(n)}_s) + r_k(B^{(n)}_{/s}) \quad \text{by} \quad (***)$$



$B^{(n)}_s$



$B^{(n)}_{/s}$

Now $r_{k-1}(B^{(n)}_s)$ is the number of ways to choose $k-1$ dots from a sequence of $2n-3$ dots, no two consecutive, and $r_k(B^{(n)}_{/s})$ is the number of ways to choose k dots, no two consecutive, from a sequence of $2n-1$ dots

<u>Lemma</u> The # of ways to choose p dots from a sequence of m dots, no two consecutive, is

$$\binom{m-p+1}{p}$$

Pf. Begin by placing p dots in a line

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x_1 \quad x_2 \quad x_3 \quad x_4 \qquad\qquad x_p \quad x_{p+1}$$

Place remaining dots in boxes, between dots or before or after the dots

sol'ns to

$$x_1 + x_2 + \cdots + x_{p+1} = m - p \qquad x_1, x_{p+1} \geq 0$$
$$x_2, x_3, \ldots, x_p \geq 1$$

= # sol'ns to $\quad x_1 + x_2 + \cdots + x_{p+1} = m - p - (p-1) \qquad x_i \geq 0$ all i

$$= \binom{m - 2p + 1 + p}{p} = \binom{m-p+1}{p}$$

Then, $r_k(B^{(n)}) = r_{k-1}(B_s^{(n)}) + r_k(B_{/s}^{(n)})$

$$= \binom{2n-3-(k-1)+1}{k-1} + \binom{2n-1-k+1}{k}$$

$$= \binom{2n-k-1}{k-1} + \binom{2n-k}{k} = \frac{(2n-k-1)!}{(k-1)!\,(2n-2k)!} + \frac{(2n-k)!}{k!\,(2n-2k)!}$$

$$= \frac{(2n-k)!}{k!\,(2n-2k)!}\left[\frac{k}{2n-k} + 1\right] = \binom{2n-k}{k}\frac{2n}{2n-k}.$$

Then

$$M_n = 2 \cdot n! \sum_{k=0}^{n} \binom{2n-k}{k} \frac{2n}{2n-k} (n-k)!\,(-1)^k \qquad \text{by } (*)$$

ex. # 8. Suppose five officials O_1, O_2, O_3, O_4, O_5 are to be assigned five different cars: an Escort, a Lexus, a Nissan, a Taurus, and a Volvo. (E, L, N, T, V)

O_1 will not drive an Escort or a Taurus

O_2 " " " a Taurus

O_3 " " " a Lexus or a Volvo

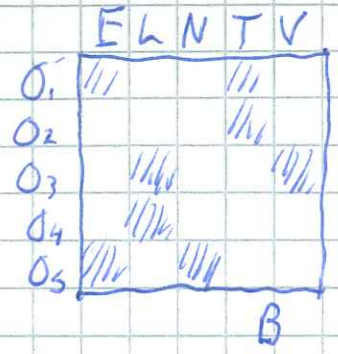
O_4 " " " a Lexus

O_5 " " " an Escort or a Nissan

If a feasible assignment of cars is chosen randomly, what is the probability that

(a) O_1 gets the Volvo?

(b) O_2 or O_5 get the Volvo?



$S = \{(O_1,E), (O_1,T), (O_5,E), (O_5,N), (O_2,T)\}$

$T = \{(O_3,L), (O_3,V), (O_4,L)\}$

(a) If O_1 gets the Volvo, then the remaining board is

we want $\boxed{\dfrac{t_0(B')}{t_0(B)}}$

3+3
+2
2+2

$R(B) = R(S)\,R(T) = (1+5x+6x^2+x^3)(1+3x+x^2)$

$= 1+8x+27x^2+24x^3+9x^4+x^5$

$t_0(B) = 5! - 8\cdot 4! + 27\cdot 3! - 24\cdot 2! + 9\cdot 1! - 0! = 20$

$R(B') = 1+5x+8x^2+4x^3$

$t_0(B') = 4! - 5\cdot 3! + 8\cdot 2! - 4\cdot 1! = 6$

$\text{Prob.} = \dfrac{t_0(B')}{t_0(B)} = \dfrac{6}{20} = \boxed{30\%}$

(b)

if O_2 gets the Volvo, remaining board is

B''

$S = \{(O_1, E), (O_1, T), (O_5, E), (O_5, N)\}$

$T = \{(O_3, N), (O_4, L)\}$

$R(S) = 1 + 4x + 3x^2 \qquad R(T) = 1 + 2x$

$R(B'') = (1 + 4x + 3x^2)(1 + 2x) = 1 + 6x + 11x^2 + 6x^3$

$t_0(B'') = 4! - 6 \cdot 3! + 11 \cdot 2! - 6 \cdot 1! = 24 - 36 + 22 - 6 = 4$

prob O_2 gets V is $\frac{4}{20} = 20\%$

if O_5 gets the Volvo, remaining board is

B'''

$R(B''') = (1 + 3x + x^2)(1 + 2x)$

$= 1 + 5x + 7x^2 + 2x^3$

$t_0(B''') = 24 - 5 \cdot 6 + 7 \cdot 2 - 2 \cdot 1 = 24 - 30 + 14 - 2 = 6$

prob O_2 or O_5 get Volvo is $\frac{4+6}{20} = 50\%$