Part 1  Graph Theory

1.1 Graph Models

\[ G = (V, E) \]

A graph \( G \) consists of vertices \( V \) and edges \( E \).

\[ V = \{1, 2, 3, 4\} \]

\[ E = \{(1, 2), (2, 4), (2, 3)\} \]

Ex. \( K_4 \) "complete graph on 4 vertices"

\[ \text{all possible edges} \]

A directed graph is a graph where each edge has a direction.

Ex. \( D \)

If edge \((a, b) \in G\), we say \( a \) and \( b \) are adjacent.
A path is a sequence of distinct vertices.

\[
P = x_1 - x_2 - \cdots - x_n, \quad \text{where} \quad x_i \text{ is adjacent to } x_{i+1}
\]
for \( 1 \leq i < n - 1 \).

If edge \( x_n - x_1 \) is there also, we have a circuit.

\[
\begin{align*}
1 & - 4 - 3 - 2 - 1 \text{ is a circuit} \\
5 & - 2 - 3 \text{ is a path}
\end{align*}
\]

If every pair of vertices is connected by some path, we say \( G \) is connected.

Ex. A connected graph \( G \). Removing edge \((1,2)\) disconnects \( G \).
A (perfect) matching is a selection of edges, no two of which share a common vertex, such that each vertex is touched by exactly one edge.

Edges correspond to jobs people can perform. Can we match people to jobs (find a perfect matching)?

B must go to c, so then.

D must go to d, but then A cannot be matched.

So no perfect matching.

A graph is bipartite if the vertices can be divided into two sets $W, Z$ such that all edges are of form $(w, z)$ where $w \in W$ and $z \in Z$. The graph above is bipartite with

$W = \{A, B, C, D, E\}$, $Z = \{a, b, c, d, e\}$
Then $G$ is bipartite iff every circuit has even length.

**Proof.** Suffices to prove for connected bipartite graphs. First assume $G$ is bipartite.

Clearly every circuit has even length. Conversely, if every circuit has even length, let $W_1$ be an arbitrary vertex, and let $W = \text{set of all vertices which can be reached from } W_1 \text{ by a path of even length}$, and $V = V \setminus W$, (set of vertices in $G$ not in $W$).

Since $G$ is connected every vertex can be reached from $W_1$ by some path. Note that if a vertex $z$ can be reached from $w_i$ by paths $P_1$ of even length and $P_2$ of odd length, the path $P_1 \cup P_2$ would be a circuit of odd length, contradiction.
Claim: there are no edges from \( u_i \) to \( u_j \) for any \( u_i \in W, u_j \in \overline{W} \). If there were,
let \( P \) be a path from \( u_i \) to \( u_j \) of even length, and \( P' \) a path from \( u_j \) to \( u_i \) of even length. Then \( P, W, \overline{W}, \text{rev}(P) \) is an odd-length circuit. Similarly, if \( \overline{Z}_i, Z_i \) were an edge from \( Z_i \) to \( Z_i \), \( Z_i, \overline{Z}_i, Z_i \), then \( \overline{Q}, Q, Q, \text{rev}(Q) \) is an even-length circuit of odd length. Hence, \( G \) is bipartite.

ex: let \( K_n \) be the complete graph on \( n \) vertices. How many edges in \( K_n \)? How many perfect matchings in \( K_n \)? Let \( K_{n,n} \) be the complete bipartite graph between two \( n \)-vertex sets. How many edges in \( K_{n,n} \)? How many perfect matchings?
# edges in $K_n = \binom{n}{2}$

# perfect matchings in $K_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-1) & \text{if } n \text{ even} \end{cases}$

A bipartite, planar graph
Then In any graph, the \# of vertices of odd degree must be even (the degree of a vertex is the number of edges incident to (i.e. touching) the vertex).

**Def.** Each edge is counted twice in the sum of all vertex degrees, i.e. \[ \sum_{v \in V} \text{deg}(v) = 2 \sum_{e \in E} 1. \]

Where \( \text{deg}(v) \) = degree of vertex \( v \).

\[ \begin{align*}
\text{deg} &= 2 \\
\text{deg} &= 3 \\
\text{deg} &= 2 \\
\text{deg} &= 1 \\
\end{align*} \]

**Defn.** An independent set of vertices is a set with no edges between them.

\[ \{a, d\} \text{ and } \{c, d\} \text{ are independent sets} \]
Two graphs $G_1(V,E)$ and $G_2(V_2,E_2)$ are isomorphic if $|V_1|=|V_2|$, $|E_1|=|E_2|$, and moreover there is a map $p: V_1 \rightarrow V_2$ such that edge $\overline{ab}$ in $G_1$ iff edge $\overline{p(a)p(b)}$ in $G_2$.

Ex.

$G_1$ and $G_2$ are not isomorphic (different # of edges).

If you add edge $\overline{23}$ to $G_2$, they become isomorphic under map $p(a)=1$, $p(c)=2$

$p(b)=4$, $p(d)=3$

$p(e)=5$

or also $p(b)=4$, $p(a)=2$

$p(e)=5$, $p(c)=3$

$p(d)=1$

and other possibilities.
A subgraph $G'$ of $G$ is a subset of vertices and edges of $G$.

Note that if $G_1$ isom. to $G_2$, then degree sets are the same.

$G_1$ deg set $(3, 3, 3, ..., 3)$ = $G_2$ deg set $(3, 3, 3, ..., 3)$

Let $\overline{G}$ = graph with same vertices as $G$, but with edge set consisting of all edges not in $G$.

Then $\overline{G}$, isom. to $\overline{G}$, iff $G_1$ isom. to $G_2$.

Ex. are $G_1$ and $G_2$ isomorphic?

$G_1, G_2$ both circuits length 7 $\Rightarrow$ $G_1, G_2$ isomorphic.
ex. # 5d are these isomorphic?

$G_1$

degrees $\{3, 4, 4, 3, 3, 4, 3\}$

neither bipartite nor cage

$G_2$

degrees $\{3, 3, 3, 4, 4, 3, 4\}$

either cage or cage

$G_1$ not isomorphic to $G_2$

In $G_2$, from any vertex we can reach any other vertex in at most 4 edges, not true for vertex a in $G_1$.
1.4 Planar Graphs

(important in electronic circuit design)

A planar graph is one that can be drawn in the plane with no edges crossing.

4 color problem: can we color the countries with 4 colors so no neighboring countries have same color? Yes - proof very long 1976 Appel + Haken

For graphs, make a vertex for each country and an edge between vertices of countries with a common border (and a vertex for unbounded region)
How many colors are needed to color the vertices so that no two vertices which are adjacent have the same color?

**Circle-Chord method to test if a graph is planar**

- find a circuit that contains all the vertices (if possible), and draw as a large circle
- remaining edges, called chords, must be drawn either inside the circle or outside the circle. Draw first chord outside the circle. Other chords are forced

![Graph example](image)

\(K_{3,3}\) not planar
ex. Is $K_4$ planar? How about $K_5$?

$K_4$ is planar

$K_5$ is not planar

Kuratowski's, 1930

A graph is planar if it does not contain a subgraph that is a $K_5$ or a $K_{3,3}$ configuration.

A $K_{3,3}$ with extra vertices added in middle of edges

Example of a $K_{3,3}$ configuration
Is the graph planar? If not, find a $K_{3,3}$ configuration inside it.

Circle chord

not planar

When choosing the first chord, it makes no difference whether you go inside or outside the circle; think of two maps drawn on Earth's surface with one having North Pole at center, and equator outside bounding circle, and the other with South Pole at center, again with equator as bounding circle. A path linking two cities in Northern Hemisphere is inside equator circle in first map and outside equator map in second one. Planarity is much easier to test than graph isomorphism.
Thm (Euler) If \( G \) is a connected planar graph, then any planar graph depiction of \( G \) has
\[
E = V + F + 2 \quad \text{regions} \quad c = |E|, \quad v = |V| \]

\[ \begin{align*}
E &= 19 \\
V &= 10 \\
F &= 11 = 19 - 10 + 2
\end{align*} \]

Ex.

By induction on the # of edges added to form \( G_n \) where \( G_n \) denotes the connected planar graph obtained after adding \( n \) edges. If \( n = 1 \),
\[
G_1 \quad \text{for which} \quad e = 1, \quad v = 3, \quad r = 1 = 1 - 2 + 2 \quad \checkmark
\]

Assume by induction that \( G_n \) satisfies \( \square \). If we add another edge between existing vertices \( x, y \) in \( G_n \),
the number of vertices stays the same, # of edges increases by 1, # of regions increases by 1, so
\( \square \) is still satisfied.
If we add a new vertex \( x \) to \( G_n \), as well as a new edge \((x, y)\), then

- \# edges \( \uparrow \uparrow \)
- \# vertices \( \uparrow \uparrow \)
- \# regions stays the same

So \( e \leq 3v - 6 \) is still satisfied.

**Corollary.** If \( G \) is a connected planar graph with \( e \geq v \), then \( e \leq 3v - 6 \).

**Pf.** Define the degree of a region to be the \# of edges one traverses when starting at a vertex on its boundary, traveling around the boundary until returning to \( v \).

Ex.

- Degree \((K) = 10\). Note edge \((x, y)\) is counted twice.

Then, region has degree \( \geq 3 \), since if boundary of region \( R \) has only 2 edges, then it is of form \( G_1 \), but parallel edges not allowed \((G_2)\), no loops either \((G_3)\) and \( e \geq 3 \) \((G_3)\).
Now each edge is counted twice when computing degree of all regions, so

\[ 2e = \sum_{R} \text{degree (R)} \geq 3r = 3(e - v + 1) \]

so \[ 2e \geq 3e - 3v + 6 \quad \text{or} \quad e \leq 3v - 6 \]

ex. Show \( K_5 \) is non-planar using \( e \leq 3v - 6 \)

For \( K_5 \), \( e = \binom{5}{2} = 10 \)

\[ v = 5 \quad 10 \leq 15 - 6 = 9 \quad \text{No} \]