Part 1 Graph Theory

1.1 Graph Models

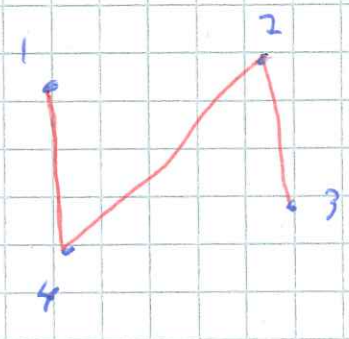
$G = (V, E)$

graph — vertices — edges

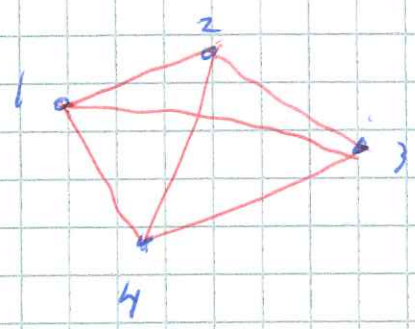
(no loops, i.e. no (a,a) edges)

ex. $V = \{1, 2, 3, 4\}$

$E = \{(1,4), (2,4), (2,3)\}$

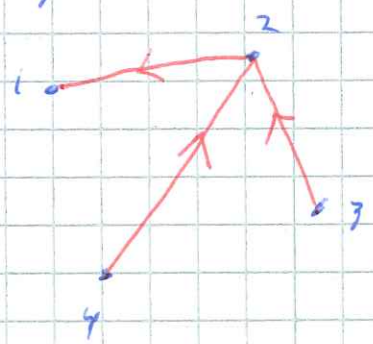


ex. K_4 "complete graph on 4 vertices"



all possible edges

A <u>directed graph</u> is a graph where each edge has a direction ex.



if edge $(a,b) \in G$, we say a and b are <u>adjacent</u>.

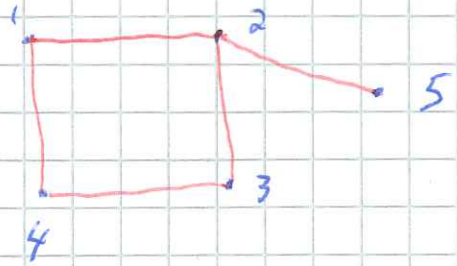
A <u>path</u> is a sequence of distinct vertices

$P = x_1 - x_2 - \dots - x_n$, where x_i is adjacent to x_{i+1}

for $1 \le i \le n-1$

If edge $x_n - x_1$ is there also, we have a <u>circuit</u>

ex

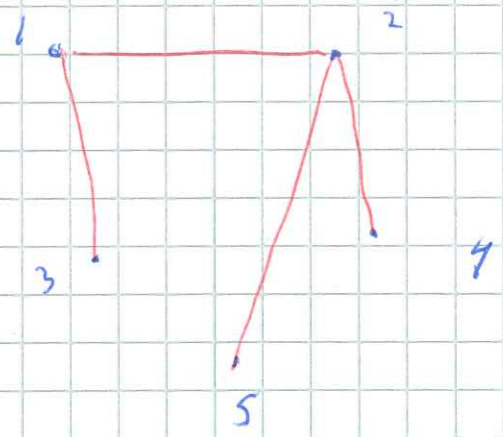


1-4-3-2-1 is a circuit

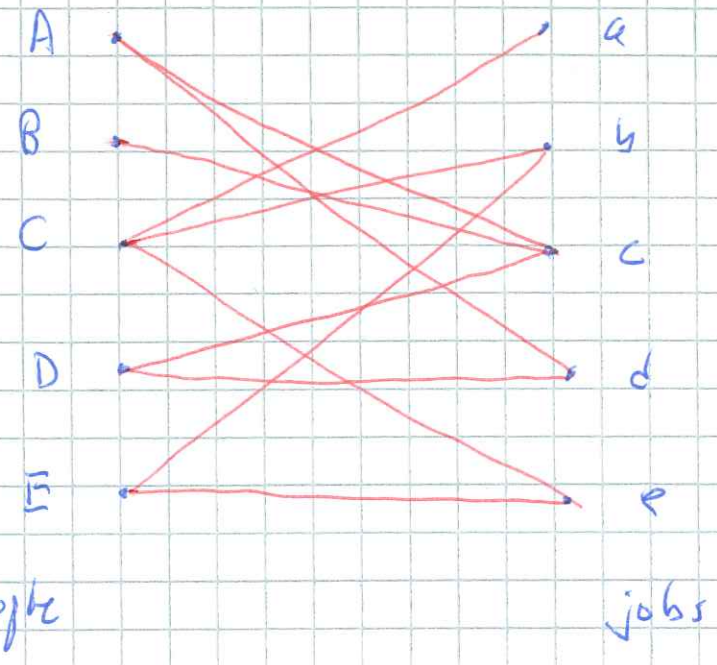
5-2-3 is a path

If every pair of vertices is connected by some path, we say G is <u>connected</u>

ex. A connected graph G. Removing edge (1,2) disconnects G



A (perfect) matching is a selection of edges, no two of which share a common vertex, such that each vertex is touched by exactly one edge



G

edges correspond to jobs people can perform. Can we match people to jobs (find a perfect matching)?

B must go to c, so then

D must go to d, but then A cannot be matched.

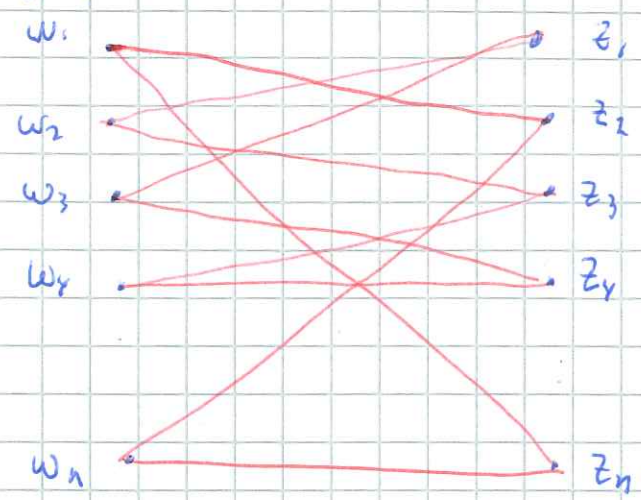
So no perfect matching.

A graph is <u>bipartite</u> if the vertices can be divided into two sets W, Z : all edges are of form (w, z) $w \in W$ $z \in Z$. The graph above is bipartite with

$W = \{A, B, C, D, E\}$, $Z = \{a, b, c, d, e\}$

<u>Thm</u> G is bipartite iff every circuit has even length.

<u>Pf.</u> Suffices to prove for connected bipartite graphs. First assume G is bipartite



Clearly every circuit has even length. Conversely, if every circuit has even length, let w_1 be an arbitrary vertex, and let W = set of all vertices which can be reached from w_1 by a path of even length, and $Z = V \setminus W$. (set of vertices in G not in W)

Since G is connected every vertex can be reached from w_1 by some path. Note that if a vertex z can be reached from w_1 by paths P_1 of even length and P_2 of odd length, the path $P_1 \, rev(P_2)$ would be a circuit of odd length, contradiction.

Claim There are no edges from w_i to w_j for any $w_i \in W$, $w_j \in W$. If there were, Let P_1 be a path from w_1 to w_i of even length, and P_2 a path from w_1 to w_j of even length. Then $P_1\, \overline{w_i w_j}\, rev(P_2)$ is an odd-length circuit. Similarly, if $\overline{z_i z_j}$ were an edge from z_i to z_j, $z_i, z_j \in Z$, Then if Q_1 were a path from w_1 to z_i of odd length, and Q_2 a path from w_1 to z_j of odd length,

$Q_1\, \overline{z_i z_j}\, rev(Q_2)$ is a circuit of odd length

(Here $rev(P)$ is the reverse of the path P).

Hence, G is bipartite

ex Let K_n be the complete graph on n vertices. How many edges in K_n? How many perfect matchings in K_n? Let $K_{n,n}$ be the complete bipartite graph between two n-vertex sets. How many edges in $K_{n,n}$? How many perfect matchings?

edges in $K_n = \binom{n}{2}$

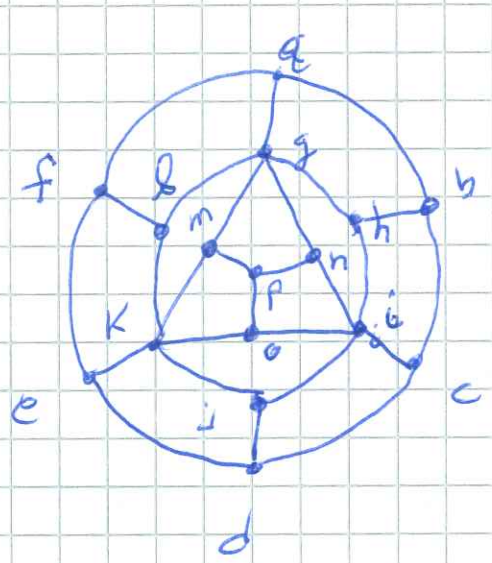
perfect matching in $K_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1 \cdot 3 \cdot 5 \cdot 7 \cdots (n-1) & \text{if } n \text{ even} \end{cases}$

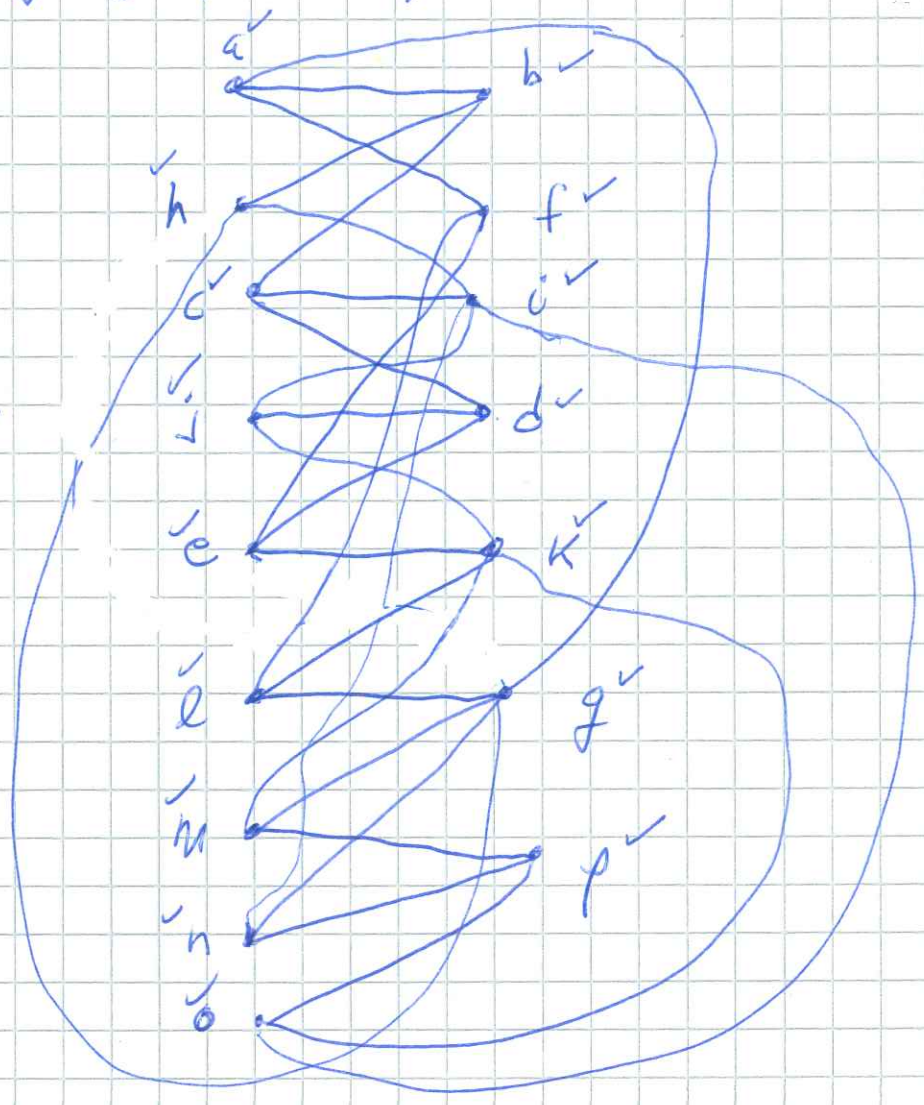
choices for edge involving vertex 1

edges in $K_{n,n} = n^2$

perfect matchings in $K_{n,n} = n!$

ex. Is the following graph bipartite?





A bipartite, planar graph

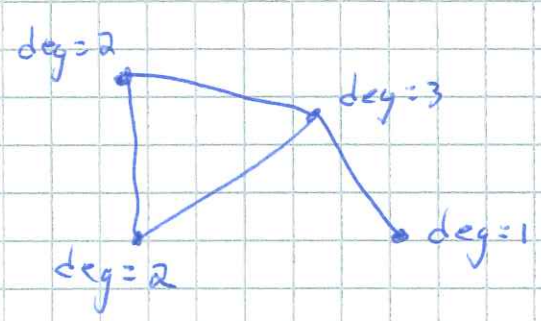
Thm In any graph, the # of vertices of odd degree must be even (The <u>degree</u> of a vertex is the number of edges <u>incident</u> to (i.e. touching) the vertex.

Pf. Each edge is counted twice in the sum of all vertice degrees, i.e.

$$\sum_{v \in V} \deg(v) = 2 \sum_{e \in E} 1.$$

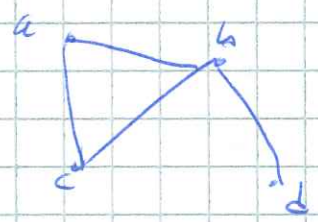
Where $\deg(v)$ = degree of vertice v.

ex.



Defn. An <u>independent</u> <u>set</u> of vertices is a set with no edges between them.

ex.

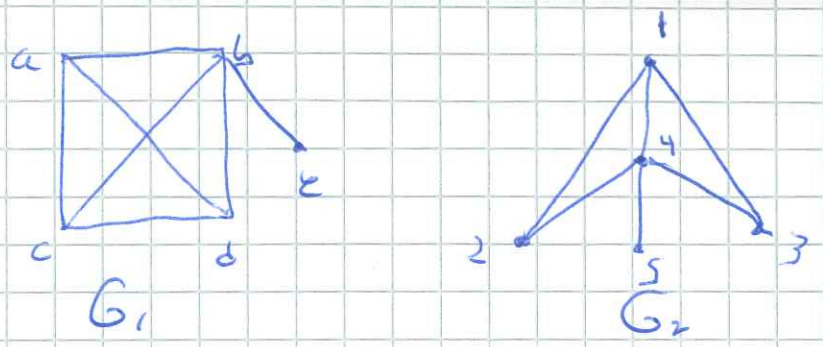


$\{a, d\}$ and $\{c, d\}$ are independent sets

Two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are <u>isomorphic</u> if $|V_1| = |V_2|$, $|E_1| = |E_2|$, and moreover there is a map $\rho : V_1 \to V_2$ such that

edge $\overline{ab}$ is in E_1 iff edge $\overline{\rho(a)\,\rho(b)} \in E_2$

ex.



are not isomorphic (different # of edges)

if you add edge $\overline{23}$ to G_2, they become isomorphic under map $\rho(a) = 1$ $\rho(c) = 2$

$\rho(b) = 4$ $\rho(d) = 3$

$\rho(e) = 5$

or also $\rho(b) = 4$ $\rho(a) = 2$

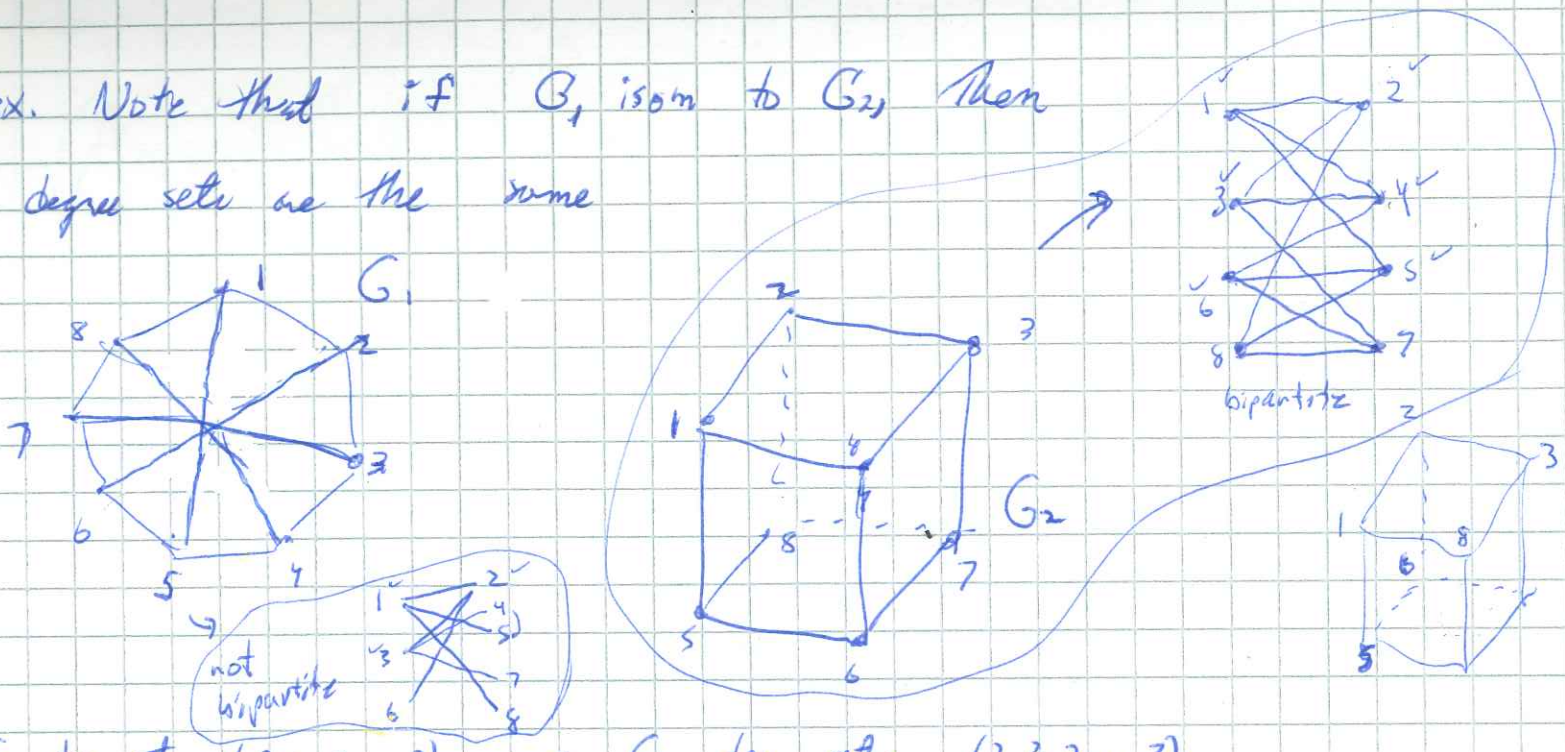
$\rho(e) = 5$ $\rho(c) = 3$

$\rho(d) = 1$

and other possibilities

A subgraph G' of G is a subset of vertices and edges of G.

ex. Note that if G_1 isom to G_2, then degree sets are the same

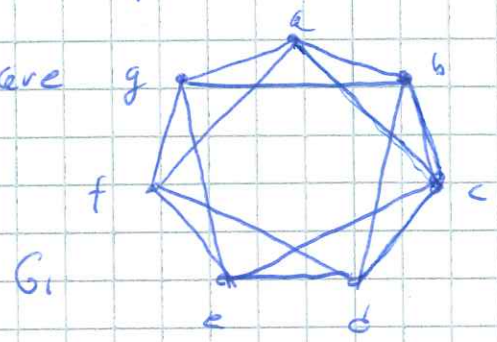


G_1 deg seq $(3,3,3,\ldots,3)$ = G_2 deg seq $(3,3,3,\ldots,3)$

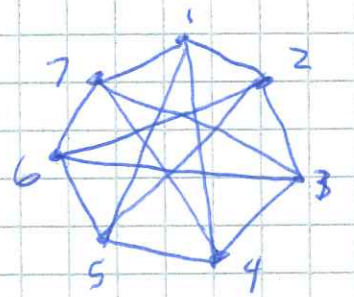
Let $\bar{G}$ = graph with same vertices as G, but with edge set consisting of all edges not in G.

<u>Thm</u> $\bar{G}_1$ isom. to $\bar{G}_2$ iff G_1 isom to G_2

ex. are

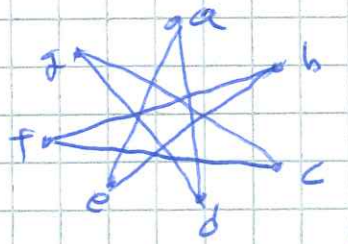


and

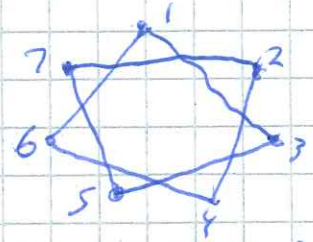


isomorphic?

$\bar{G}_1 =$

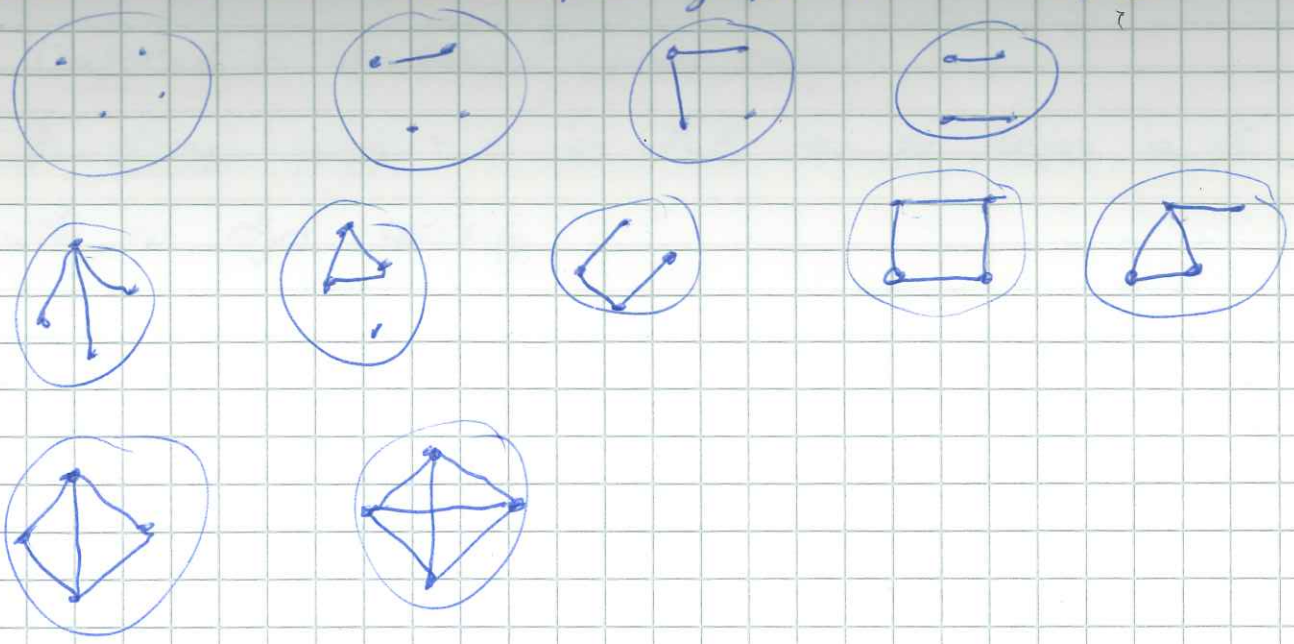


$\bar{G}_2 =$

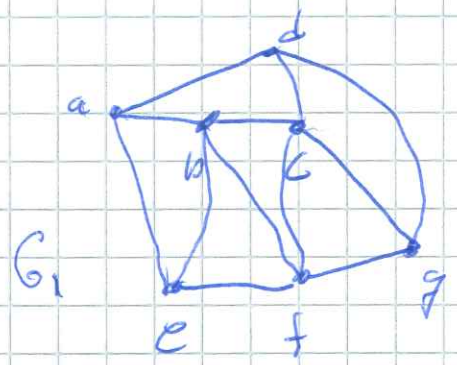


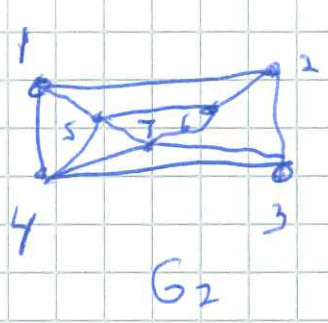
$\bar{G}_1, \bar{G}_2$ both circuits length 7 $\Rightarrow$ G_1, G_2 isomorphic

ex. #1 List all non-isomorphic graphs on 4 vertices



ex. # 5d are these isomorphic?





degrees (3, 4, 4, 3, 3, 4, 3)

degrees (3, 3, 3, 4, 4, 3, 4)

neither bipartite since a, b, e & 1, 5, 7 are cycles odd length

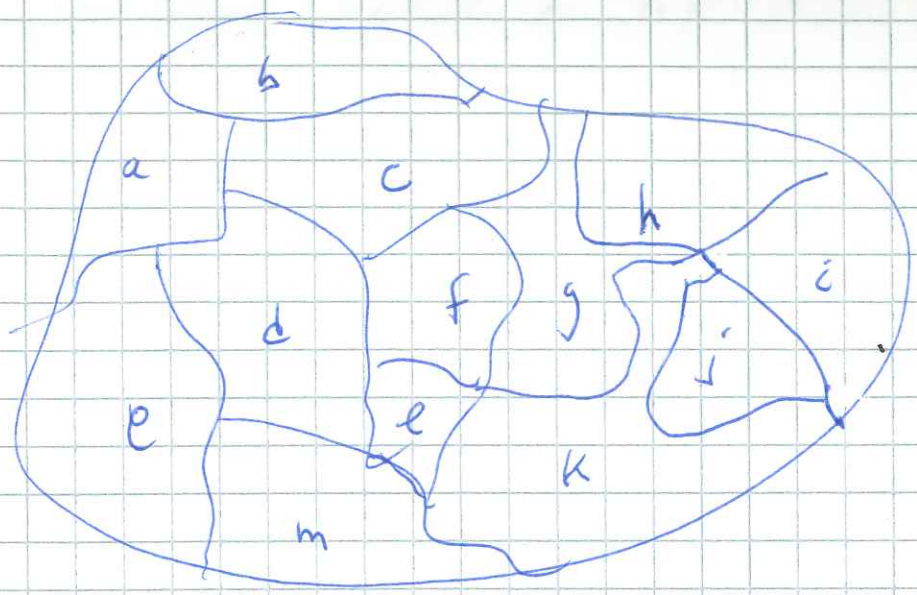
In G_2, from any vertex we can reach any other vertex in at most 2 edges, not true for vertex a to g in G_1 (in at least two different ways)

G_1 not isom. to G_2

1.4 Planar Graphs

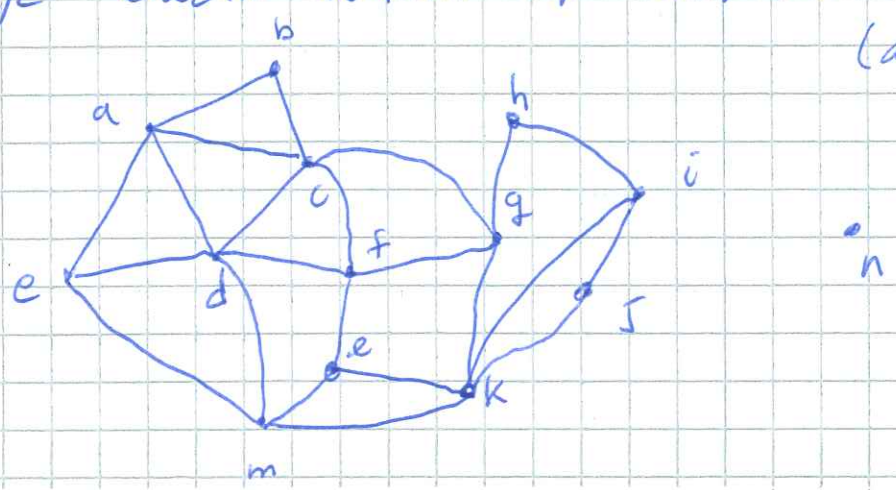
(important in electronic circuit design)

A <u>planar</u> graph is one that can be drawn in the plane with no edges crossing



4 color problem: can we color the countries with 4 colors so no neighboring countries have same color? Yes - proof very long 1976 Appel + Haken

Dual graph: make a vertex for each country and an edge between vertices of countries with a common border (and a vertex for unbounded region)



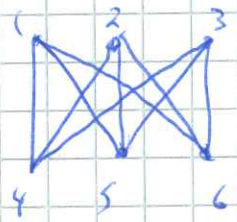
meeting in a point doesn't count

How many colors are needed to color the vertices so that no two vertices which are adjacent have the same color?

<u>Circle-Chord</u> <u>method</u> to test if a graph is planar

- find a circuit that contains all the vertices (if possible), and draw as a large circle
- remaining edges, called chords, must be drawn either inside the circle or outside the circle. Draw first chord outside the circle. Other chords are forced.

ex.



$K_{3,3}$

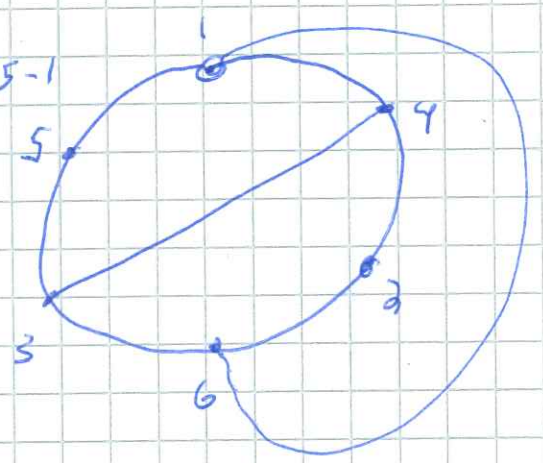
Circuit 1-4-2-6-3-5-1

add 1-6

then 3-4

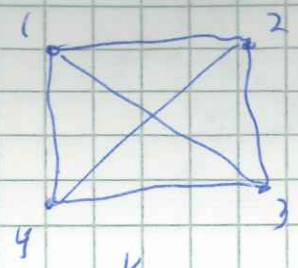
now 2-5

can't

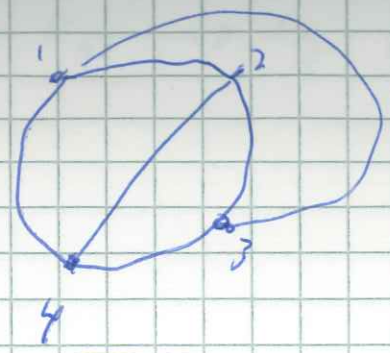


K_{33} not planar

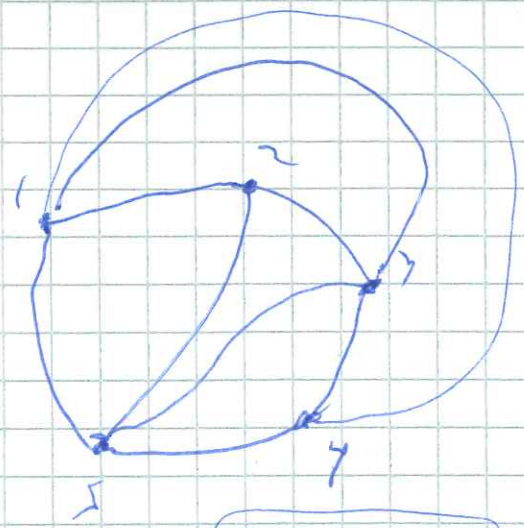
ex. Is K_4 planar? How about K_5?



K_4



K_4 is planar



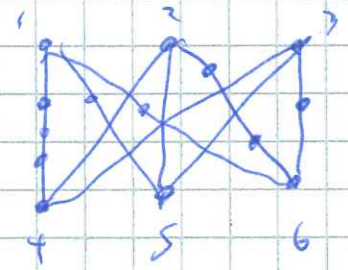
K_5 not planar

can't add 2y

Kuratowski, 1930

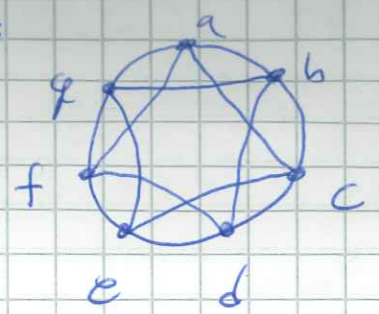
Thm A graph is planar iff it does not contain a subgraph that is a K_5 or a $K_{3,3}$ configuration

a K_{33} with extra vertices added in middle of edges



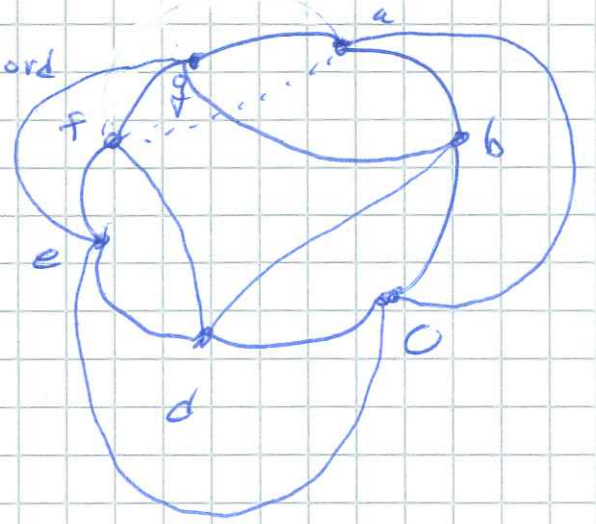
example of a $K_{3,3}$ configuration

ex. Is

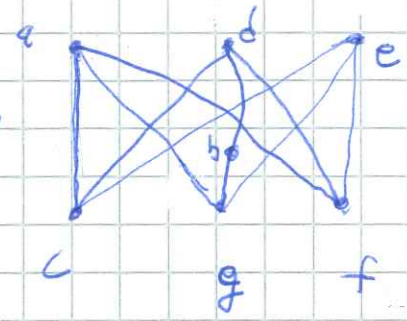


planar? If not, find a $K_{3,3}$ config. inside it

circle-chord



not planar



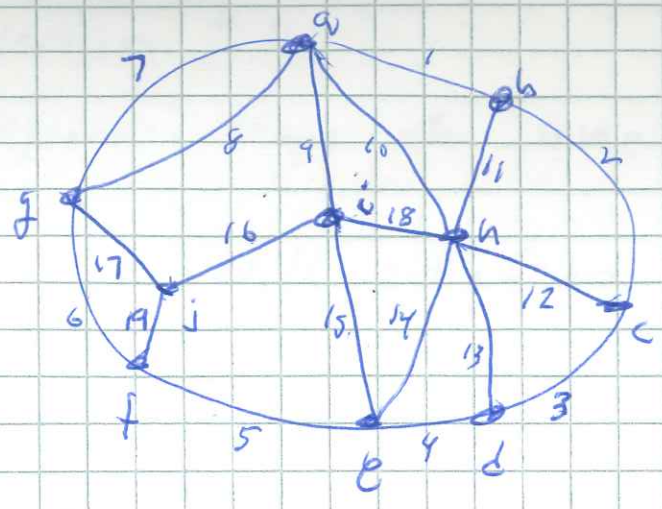
$K_{3,3}$ config.

- When choosing 1st chord, it makes no difference whether you go inside or outside the circle; think of 2 maps drawn on Earth's surface with one having North Pole at center, and equator outside bounding circle, and the other with the South pole at center, again with equator as bounding circle. A path linking 2 cities in Northern Hemisphere is inside equator circle in 1st map and outside equator map in second one.
- planarity is much easier to test than graph isomorphism

Thm (Euler) If G is a connected planar graph, then any plane graph depiction of G has

(*) $r = e - v + 2$ regions $\quad e = |E|, \; v = |V|$

ex.



ex.

$3 - 3 + 2 = 2$ ✓

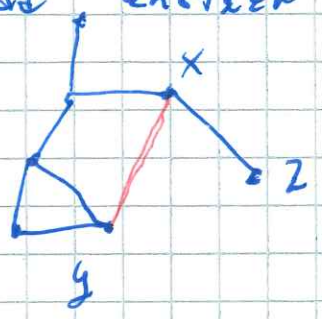
$e = 19 \qquad r = 11 = 19 - 10 + 2$

$v = 10$

Pf. By induction on the # of edges added to form G_n where G_n denotes the connected planar graph obtained after adding n edges. If $n=1$,

$G_1 =$  , for which $e=1, v=2, r=1 = 1-2+2$ ✓

Assume by induction that G_{n-1} satisfies (*). If we add another edge between existing vertices x, y in G_{n-1}



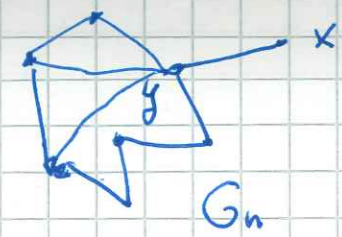
the numbers of vertices stays the same, # of edges increases by 1, # of regions increases by 1, so (*) is still satisfied.

If we add a new vertex x to G_{n-1} as well as a new edge (x,y), then

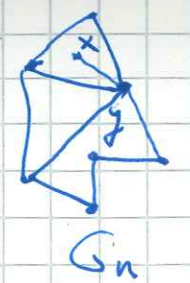
edges $\uparrow 1$

vertices $\uparrow 1$

regions stays the same



or

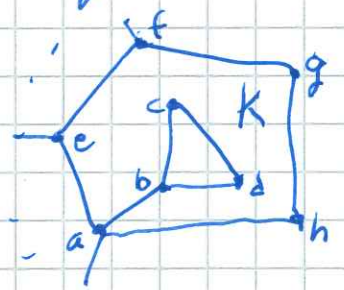


so (*) is still satisfied

<u>Corollary</u> If G is a connected planar graph with $e > 1$, then $e \le 3v - 6$

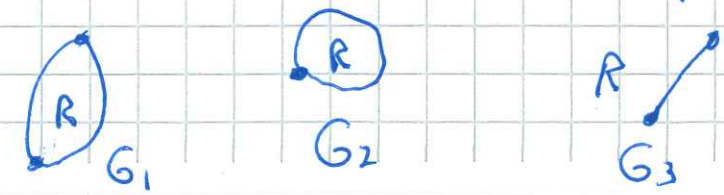
<u>Pf.</u> Define the degree of a region to be the # of edges one traverses when starting at a vertex w on its boundary, travelling around the boundary until returning to w.

ex.



degree$(K) = 10$. Note edge (x,y) is counted twice

Then region has degree ≥ 3, since if boundary of region R has only 2 edges, then it is of form



but parallel edges not allowed (G_1), no loops either (G_2) and $e > 1$ (G_3).

Now each edge is counted twice when computing degree of all regions, so

$$2e = \sum_{\substack{\text{regions} \\ R}} \text{degree}(R) \geq 3r = 3(e - v + 2)$$

so $2e \geq 3e - 3v + 6$ or $\boxed{e \leq 3v - 6}$

ex. Show K_5 is nonplanar using $e \leq 3v - 6$

For K_5, $e = \binom{5}{2} = 10$

$v = 5$ $\quad 10 \leq 15 - 6 = 9$ $\quad$ <u>No</u>