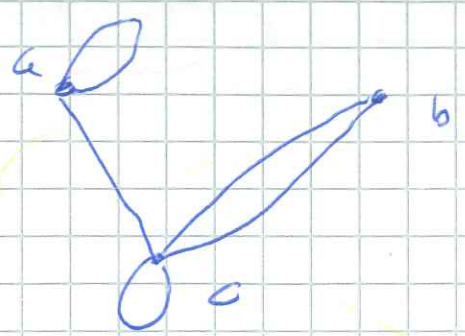
Chapter 2 Covering Circuits and Graph Coloring

Multigraph: Multiple edges between vertices,
loops also also web

ex

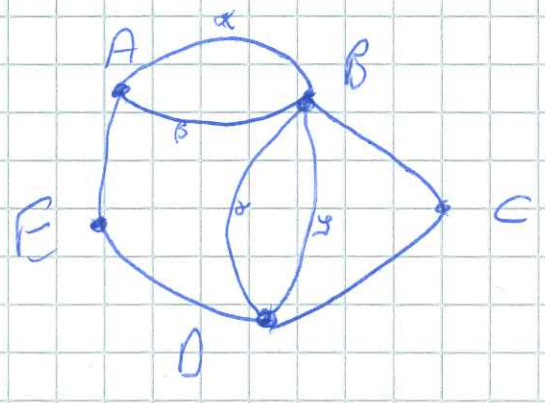


recall path $P = x_1 - x_2 - \cdots - x_n$ sequence distinct vertices joined by edges

- if also edge $x_n - x_1$, we get a circuit

trail $T = x_1 - x_2 - \cdots - x_n$ vertices may not be distinct (but edges cannot be repeated)

ex



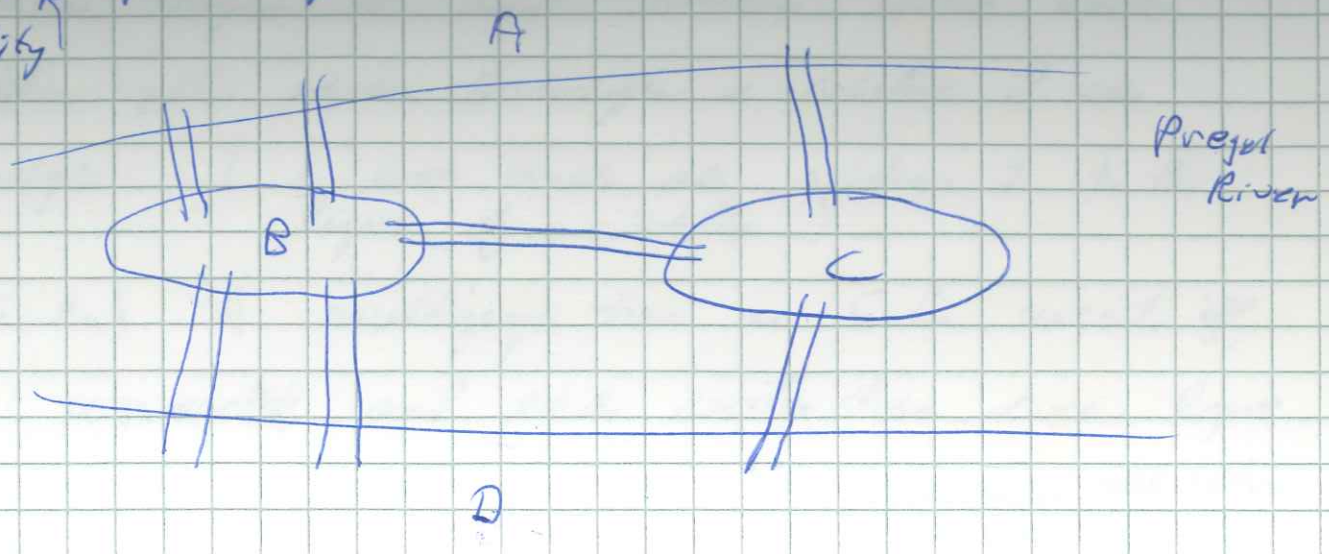
$A \overset{\alpha}{-} B \overset{\delta}{-} D \overset{\gamma}{-} B - C$

trail

- if edge $x_n - x_1$ is also added to T, T becomes a <u>cycle</u> so $A \overset{\alpha}{-} B \overset{\delta}{-} D \overset{\gamma}{-} B \overset{\beta}{-} A$ is a cycle

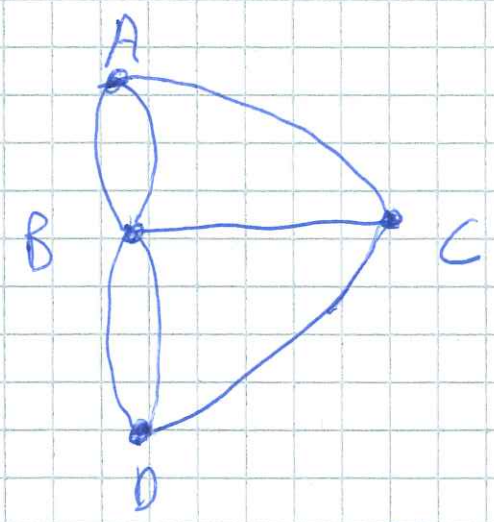
Konigsberg Bridge Problem

Prussian City



Can a townsperson walk from A and back again crossing each of the 7 bridges exactly once?

Euler



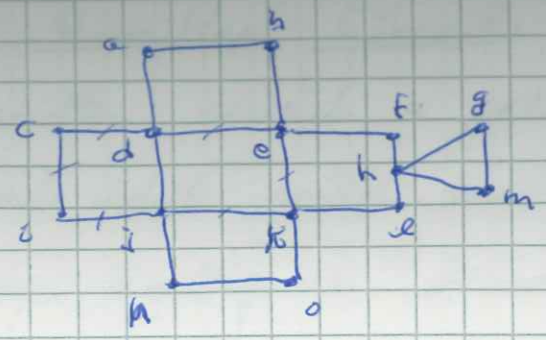
can be find a cycle from A to A using each edge once? An Euler Cycle is a cycle using each edge once and which visits each vertex

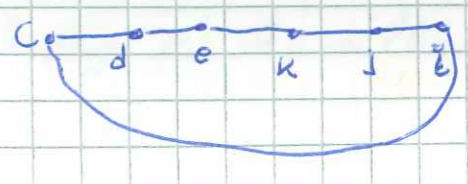
• A multigraph with an Euler cycle will have an even degree at each vertex since each time the cycle passes through a vertex it uses two edges (a loop counts as adding 2 to the degree of a vertex)

Thm (Euler) A multigraph has an Euler circuit iff it is connected and each vertex has even degree

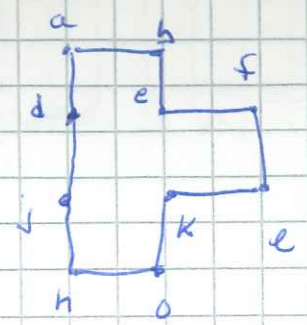
Algorithm to construct an Euler cycle: (assuming all vertices even degree) Start at any vertex A and trace out a trail. By even degree condition, we are never forced to stop at any vertex unless it is A, so our trail eventually ends at A. Let C = cycle thus generated, and G' = G with edges of C removed. G' may not be connected but all vertices are of even degree. Note C and G' must have a common vertex or no path from A to vertices in G'. Let a' = common vertex. When trail reaches a', add side trail consisting of Euler cycle for connected component of G' starting and ending at a'. ∎

Ex.





$G' =$



if d = common vertex

so cdabefekonjdekjic is an Euler Cycle

Thm A multigraph has an Euler trail, but not an Euler cycle, iff it has exactly 2 vertices of odd degree (and is connected)

Pf ⇒ Suppose G has an Euler trail T. Then the starting and ending vertices have odd degree while other vertices have even degree, and G is connected.

⇐ Suppose G has exactly 2 vertices p, q of odd degree and is connected. Add an extra edge $\overline{pq}$ to G to obtain G'. Then G' has an Euler circuit, and removing $\overline{pq}$ from this yields an Euler trail in G. ∎

2.2 Hamilton Circuits

Hamilton circuits + paths are circuits and paths which visit each vertex exactly once

ex. Routing a delivery truck which must visit a set of stores.

- finding a Hamilton circuit or path is an NP-complete problem

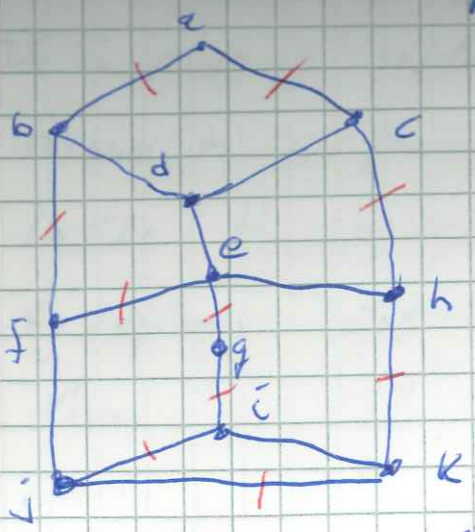
Note: If a graph has a Hamilton circuit, then any such Hamilton circuit must contain exactly 2 edges incident to each vertex. Furthermore

Rule 1 If a vertex has degree 2, both of the edges incident to x must be part of any Hamilton circuit

Rule 2 No proper subcircuit - that is, a circuit not containing all vertices - can be formed when building a Hamilton circuit

Rule 3 Once the Hamilton circuit is required to use two given edges at a vertex x, all other (unused) edges incident to x must be removed from consideration

ex Show this graph has no Hamilton Circuit



- By Rule 1, b-a-c and e-g-i must be part of the circuit

Now consider vertex i; g-i must be part of circuit, so i-j or i-k must be (but not both). By symmetry, choose i-j & remove edge i-k.

then j-k and k-h must be in circuit, so remove f-j; this forces b-f and e-f to be in circuit, remove d-e & e-h, force c-h to be in circuit, remove c-d and b-d; No edge left to get to d.

<u>Thm.</u> (Dirac, 1952) A graph with n vertices, $n > 2$, has a Hamilton circuit if the degree of each vertex is $\geq n/2$

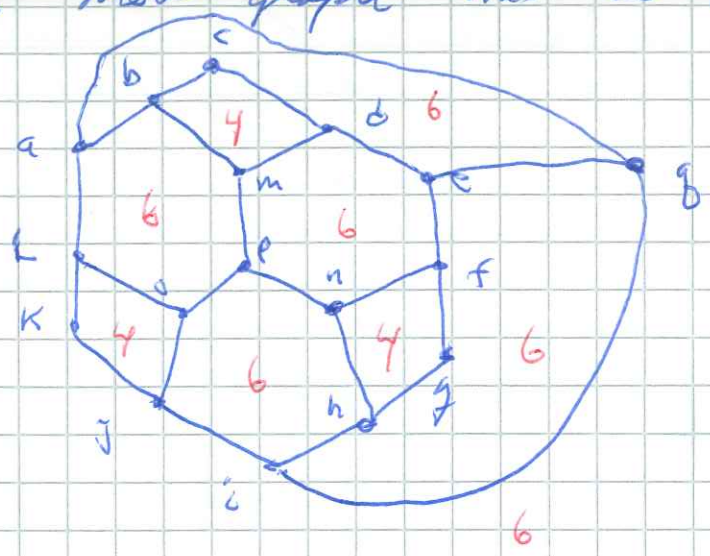
<u>Thm.</u> (Chvatal, 1972) Let G be a connected graph with n vertices, and let the vertices be indexed $x_1, x_2, \ldots, x_n$, so that $\deg(x_i) \leq \deg(x_{i+1})$. If for each $k \leq n/2$, either $\deg(x_k) > k$ or $\deg(x_{n-k}) \geq n-k$, then G has a Hamilton circuit

Thm (Grinberg, 1968)

If G planar with Hamilton circuit H, let G be drawn in a planar way, and let r_i = # regions inside H bounded by i edges, and let r_i' = # regions outside the circuit bounded by i edges, then

$$\sum_i (i-2)(r_i - r_i') = 0$$

ex. Show graph has no Hamilton circuit



$r_4 + r_4' = 3$

$r_6 + r_6' = 6$

$2(r_4 - r_4') + 4(r_6 - r_6') = 0$

or $r_4 - r_4' + 2r_6 - 2r_6' = 0$

Now $r_4 - r_4'$ must be odd since $r_4 + r_4' = 3$. But $2r_6 - 2r_6'$ is even.

<u>Defn</u> A complete graph, where every edge is given a direction, is called a <u>tournament</u>

<u>Thm.</u> Every tournament has a Hamilton path

Pf. Induction on # vertices. Trivially true for $n=2$.

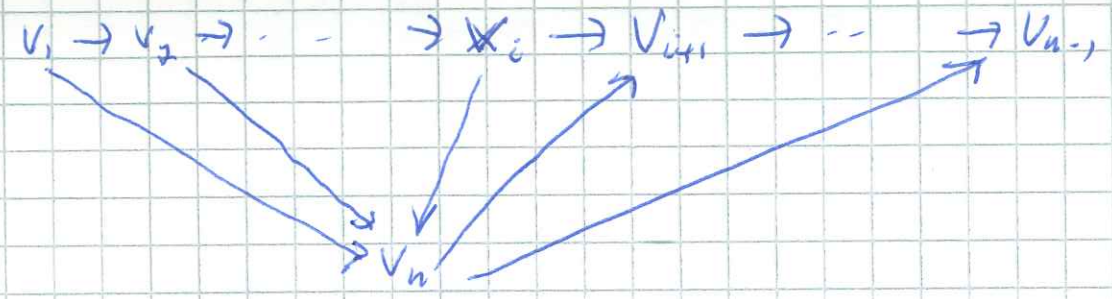
Assume true for $n-1$, $n \geq 3$.

Remove vertex n from G & find Hamilton path

$v_1 \to v_2 \to \quad \to v_{n-1}$ in G'

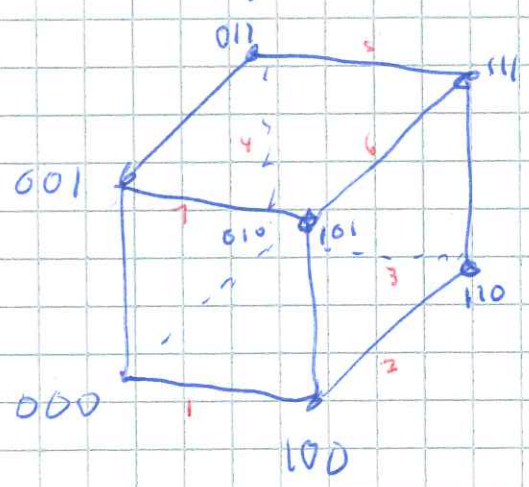
if $v_n \to v_1$ or $v_{n-1} \to v_n$ done. Else

$v_1 \to v_n$ and $v_n \to v_{n-1}$.



new path $v_1 \to v_2 \to \cdots \to v_i \to v_n \to v_{i+1} \to \cdots \to v_{n-1}$

ex. Gray Code: list elements of a set of elements where consecutive members of the list are as similar as possible, for example, subsets of $\{1,2,3\}$



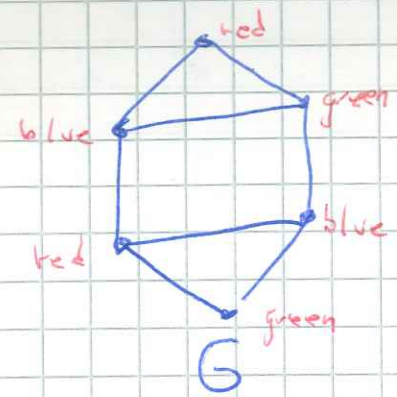
Find a Hamilton Path in this graph

000 - 100 - 110 - 010 - 011 - 111 - 101 - 001

2.3 Graph Coloring

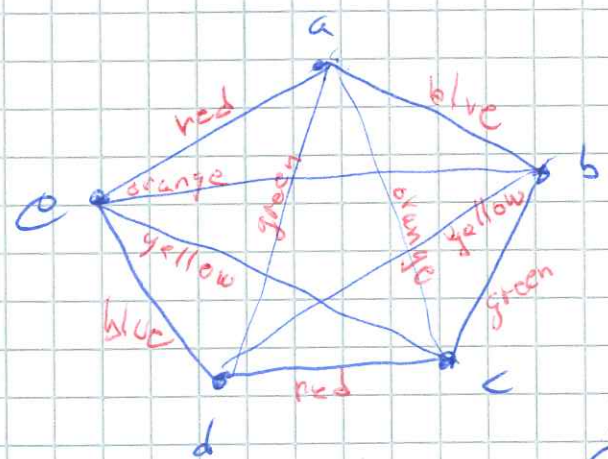
- Assign colors to vertices so adjacent vertices have different colors. <u>Chromatic Number</u> = minimum # of colors needed for a given graph G, denoted $\chi(G)$.
- NP-complete to find $\chi(G)$ (and prove it)

ex.



$\chi(G) = 3$

ex. Round Robin Tournament (each of n contestants plays each other). Schedule → each player can play at most once per day. What is minimum # of days?



edges represent games to be played.

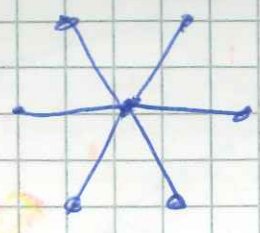
Model by edge-coloring problem (edges with common vertex have different colors)

Days ~ Colors

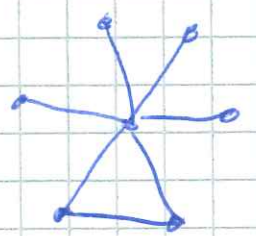
Since at most 2 games on any given day, and 10 edges, we need at least 5 colors

ex. Find a graph on 7 vertices which is

- planar
- 3-chromatic
- No Euler cycle

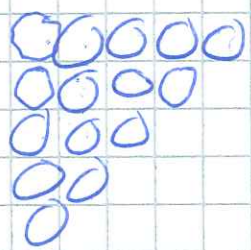


2-chromatic no Euler cycle: Add an edge to get

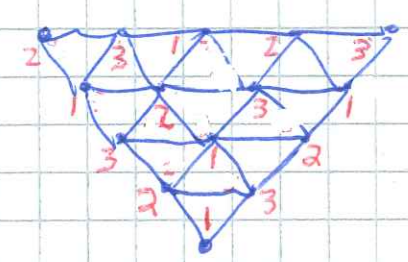


Now its 3-chromatic

ex.# 11 How many colors are needed to color the 15 billiard balls so that touching balls are different colors?



some vertices are adjacent to 6 others, so need at least 3 colors needed.

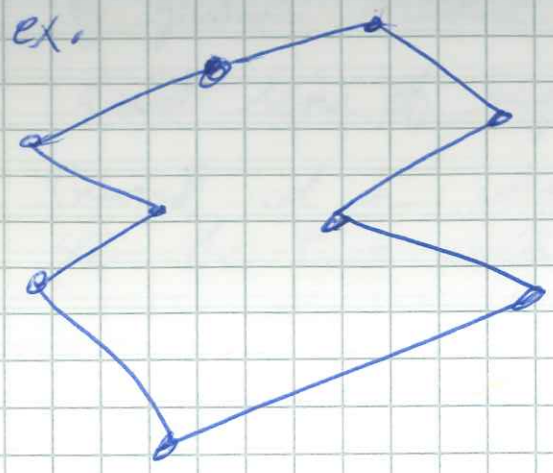


a three coloring

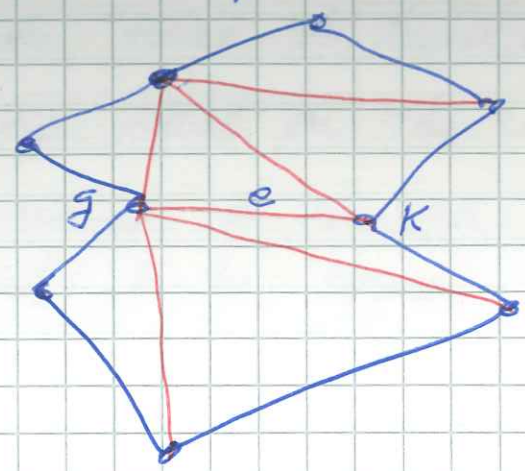
2.4 Coloring Theorems

A polygon is a plane graph consisting of a single circuit with edges drawn in straight lines

ex.



A polygon



A triangulation
(add non-crossing chords between vertices)

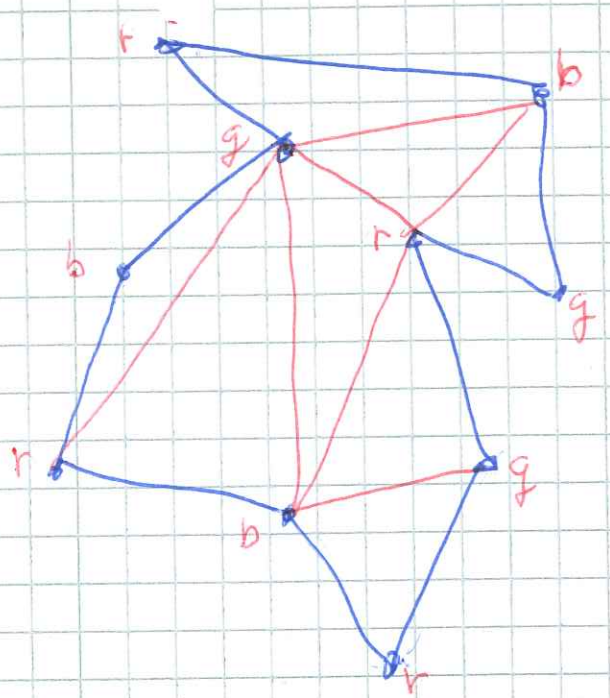
Thm A triangulation of a polygon can be 3 colored

Pf. By induction on n, the # of vertices (= # edges in polygon circuit)

$n=3$ clear. Let T be a polygon with n edges. Pick a chord edge e, e.g. chord (g,k) in the Figure above. Note T must have at least one chord edge, since $n \geq 4$. Chord e splits T into 2 smaller polygons; by induction each can be triangulated. Color both, choosing colors so edge e has same colors at end vertices (relabelling if needed). ∎

Art Gallery Problem

- place guards so that all the walls of an art gallery are watched. What is the minimum # of guards needed? Guards need to have a direct line of sight to every point on the walls. Guard at a corner is assumed to be able to see the two walls that end at the corner.



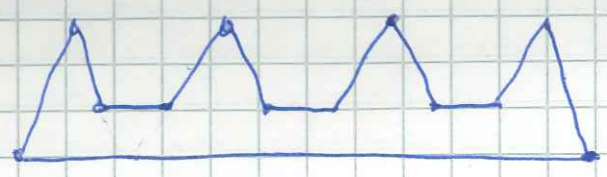
Thm The Art Gallery Problem with n walls requires at most $\lfloor n/3 \rfloor$ guards

Pf. Triangulate + color with red, blue, and green.

Place guard at each red vertex - this guard sees all sides of the triangle, so now each wall is viewed by

a guard. If there are n walls, then some color is used at most $\lfloor n/3 \rfloor$ times

example where $\lfloor n/3 \rfloor$ is needed;



12 edges Clearly need at least 4 guards

<u>Thm.</u> If a graph G is not an odd circuit or a complete graph, then $\chi(G) \leq d$, where d is the maximum degree of a vertex of G.

<u>Def'n</u> Edge chromatic # = minimum # of colors needed to color edges so edges with common end vertex get different colors.

<u>Thm 4</u> If max degree of a vertex in G is d, then the edge chromatic # of G is either d or $d+1$

<u>Thm</u> Every planar graph can be 5-colored (can actually be 4-colored but proof very long)

Pf. See book pages 80-81

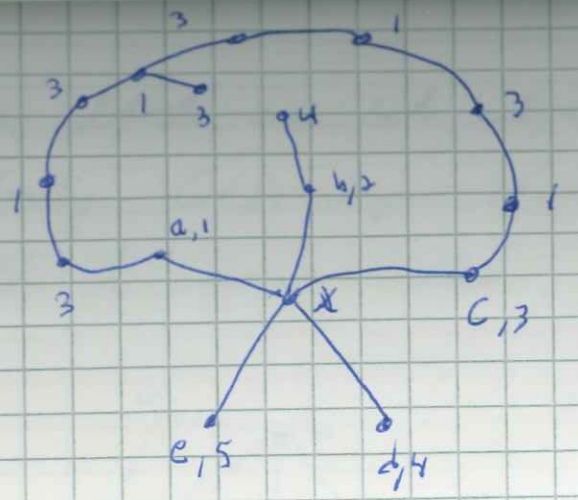


Figure used in proof of Thm. 4

Fact used in proof of Thm. 4. Any planar graph has a vertex of degree ≤ 5

Pf. Assume not. Then $\sum_{v \in G} \deg(v) \geq 6v$

But $\sum_{v \in G} \deg(v) = 2e$ so $2e \geq 6v$

$e \geq \frac{6}{2}v = 3v$

But $e \leq 3v-6$

so $3v-6 \geq e \geq 3v \Rightarrow 3v-6 \geq 3v$ contradiction

Chromatic Polynomial

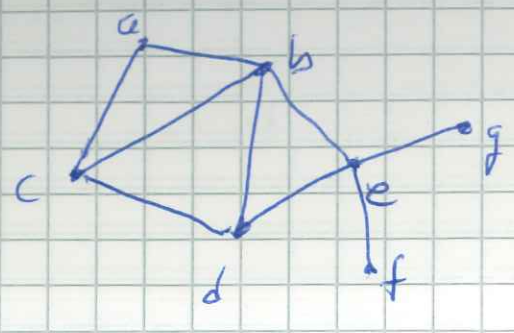
Def'n $P_K(G) =$ # of ways to color a graph G with K colors.

Thm. $P_K(G)$ is a polynomial in K.

Pf. Given any coloring C of G, form a set partition $\lambda(C)$ by placing vertices with the same color in the same block. Conversely, given a set partition λ of G with the property that two vertices in the

same block are not adjacent, one can form a coloring $C(\lambda)$ by coloring vertices in the same block with the same color.

ex.



$\lambda = \{a, d\}\ \{c, e\}\ \{b, f, g\}$

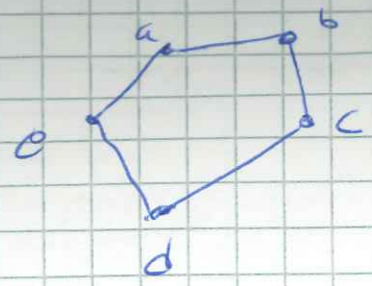
For each such λ with m blocks, there are $k(k-1)\cdots(k-m+1)$ ways to color the blocks with k colors. Thus

$$P_k(G) = \sum_{\substack{\lambda \\ m = \#\text{blocks} \\ \text{of } \lambda}} k(k-1)\cdots(k-m+1)$$

$$= \sum_{m} k(k-1)\cdots(k-m+1) \sum_{\substack{\lambda \\ m \text{ blocks}}} 1$$

ex. Find $P_k(G)$

$G =$



λ: 3 blocks

$\{a,c\}$ $\{be\}$ $\{d\}$

$\{a,c\}$ $\{bd\}$ $\{e\}$

$\{ad\}$ $\{be\}$ $\{c\}$

$\{ad\}$ $\{ce\}$ $\{b\}$

$\{a\}$ $\{bd\}$ $\{ce\}$

λ: 4 blocks

$\{ae\}$ $\{e\}$ $\{d\}$ $\{b\}$ $\qquad$ $\{a\}$ $\{ec\}$ $\{b\}$ $\{d\}$

$\{ad\}$ $\{c\}$ $\{b\}$ $\{e\}$ $\qquad$ $\{a\}$ $\{e\}$ $\{bd\}$ $\{c\}$

$\{a\}$ $\{eb\}$ $\{c\}$ $\{d\}$

$\lambda = 5$ blocks $\qquad$ $\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$ $\{e\}$

$$P_k(G) = k(k-1)(k-2)\,5 + k(k-1)(k-2)(k-3)\,5 + k(k-1)(k-2)(k-3)(k-4)$$

ex. A connected graph with no circuits is called a <u>tree</u>. If T is a tree with n vertices, then $P_k(T) = k(k-1)^{n-1}$