Chapter 3
Covering Circuits and Graph Coloring

Multigraph: Multiple edges between vertices, loops also allowed.

Example:

Correct path $P = x_1 - x_2 - \cdots - x_n$

If a loop edge $x_n - x_i$ is added, we get a circuit.

Trail $T = x_i - x_m - \cdots - x_n$

Vertices may not be distinct (but edges cannot be repeated).

Sequence distinct vertices joined by edges.

Example:

$A \xrightarrow{k} B \xrightarrow{5} D \xrightarrow{r} B \xrightarrow{3} C$

Trail $T$ becomes a cycle.

So $A \xrightarrow{k} B \xrightarrow{5} D \xrightarrow{r} B \xrightarrow{3} A$

is a cycle.
Königsberg Bridge Problem

Can a towns person walk from A and back again crossing each of the 7 bridges exactly once?

Euler

Can we find a cycle from A to A using each edge once? An Euler cycle is a cycle using each edge once and which visits each vertex once.

Preger River
A multigraph with an Euler cycle will have an even degree at each vertex; since each time the cycle passes through a vertex, it uses two edges (or loop counts and adding 2 to the degree of a vertex).

Then (Euler) A multigraph has an Euler circuit if it is connected and each vertex has even degree (assuming all vertices.

Algorithm to construct an Euler cycle: Start at any vertex A and trace out a trail. By even degree condition, we are never forced to stop at any vertex unless it is A, so our trail eventually ends at A. Let C = cycle thus generated, and G' = G with edges of C removed. G' may not be connected, but all vertices are of even degree. Note C and G' must have a common vertex or no path from A to vertex in G'. Let a' = common vertex. When trail reaches a', add side trail consisting of Euler cycle for connected component of G' starting and ending at a'.
A multigraph has an Euler trail, but not an Euler cycle, if it has exactly 2 vertices of odd degree (and is connected).

If \( G \) has an Euler trail, then the starting and ending vertex have odd degree, while other vertices have even degree, and \( G \) is connected.

Let \( G \) have exactly 2 vertices of odd degree, and \( G \) is connected. Add an extra edge \( F \) to \( G \) to obtain \( G' \). Then \( G' \) has an Euler circuit, and removing \( F \) from this yields an Euler trail in \( G \). \( \square \)
2.2 Hamilton Circuits

Hamilton circuits or paths are circuits and paths which visit each vertex exactly once.

Ex. Routing a delivery truck which must visit a set of stores.

Finding a Hamilton circuit or path is an NP-complete problem.

Note: If a graph has a Hamilton circuit, then any such Hamilton circuit must contain exactly 2 edges incident to each vertex. Furthermore:

Rule 1: If a vertex has degree 2, both of the edges incident to that vertex must be part of any Hamilton circuit.

Rule 2: No proper sub-circuit - that is, a circuit not containing all vertices - can be formed when building a Hamilton circuit.

Rule 3: Once the Hamilton circuit is required to use two given edges at a vertex x, all other (unused) edges incident to x must be removed from consideration.
Show this graph has no Hamilton circuit.

By Rule 1, b-a-c and e-f-i must be part of the circuit. Now consider vertex i; g-i must be part of circuit, so i-f or i-g must be (but not both). By symmetry, choose i-j or remove edge i-k.

Then j-k and k-h must be in circuit, so remove f-j; this forces b-f and e-f to be in circuit, remove d-e and e-h, forces c-h to be in circuit, remove c-d and b-d; no edge left to get to i.

Thm. (Dirac, 1952) A graph with \( n \) vertices, \( n \geq 3 \), has a Hamilton circuit if the degree of each vertex is \( \geq \frac{n}{2} \).

Thm. (Chvátal, 1972) Let \( G \) be a connected graph with \( n \) vertices, and let the vertices be indexed \( x_1, x_2, \ldots, x_n \), so that \( \deg(x_i) \leq \deg(x_{i+1}) \). If for each \( k \leq \frac{n}{2} \), either \( \deg(x_k) > k \) or \( \deg(x_{n-k}) > n-k \), then \( G \) has a Hamilton circuit.
If $G$ is planar with Hamilton circuit $H$, let $S$ be drawn in a plane way, and let $r_i = \#$ regions inside $H$ bounded by $i$ edges, and let $r_i' = \#$ regions outside the circuit bounded by $i$ edges. Then $\sum (i-2)(r_i - r_i') = 0$.

Show graph has no Hamilton circuit

\[
\begin{align*}
r_4 + r_4' &= 3 \\
r_6 + r_6' &= 6 \\
2(r_4 - r_4') + 4(r_6 - r_6') &= 0
\end{align*}
\]

or $r_4 - r_4' + 2r_6 - 2r_6' = 0$

Now $r_4 - r_4'$ must be odd since $r_4 + r_4' = 3$. But $2r_6 - 2r_6'$ is even,

Def. A complete graph, where every edge is given a direction, is called a tournament. Then every tournament has a Hamilton path (cf. Induction on $\#$ vertices. Trivially true for $n=2$.}
Assume true for \( n-1, \ n \geq 3 \).

Remove vertex \( n \) from \( G \) to find Hamilton path

\[ v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_n \]

If \( v_{n-1} \rightarrow v_n \) done Else

\[ v_1 \rightarrow v_{n-1} \text{ and } v_n \rightarrow v_{n-1} \]

New path

\[ v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_i \rightarrow v_{n-1} \rightarrow v_n \rightarrow v_{n-1} \rightarrow \cdots \]

**Ex. Gray Code:** list elements of a set of elements where consecutive members of the list are as similar as possible. For example, subsets of \( \{1, 2, 3\} \)

Find a Hamilton Path in

This graph

\[ 000 \rightarrow 100 \rightarrow 110 \rightarrow 010 \rightarrow 011 \rightarrow 111 \rightarrow 101 \rightarrow 000 \]
2.3 Graph Coloring

- Assign colors to vertices so adjacent vertices have different colors. **Chromatic Number** = minimum # of colors needed for a given graph G, denoted \( \chi(G) \).
- NP-complete to find \( \chi(G) \) (and prove it)

ex.

![Graph]

\[ \chi(G) = 3 \]

ex. Round Robin Tournament (each of n contestants plays each other). Schedule – each player can play at most once per day. What is minimum # of days?

Model by edge-coloring problem (edges with common vertex have different colors)

Days = colors
Since at most 2 games on any given day, and 10 edges, we need at least 5 colors.
ex. Find a graph on 7 vertices which is:
1. planar
2. 3-chromatic
3. No Euler cycle

3-chromatic no Euler cycle. Add an edge to get

Now its 3-chromatic.

ex. 11: How many colors are needed to color the 15 billiard balls so that touching balls are different colors?

Some vertices are adjacent to 6 others, so need at least 3 colors needed.

A three coloring.
2.4 Coloring Theorems

A polygon is a plane graph consisting of a single circuit with edges drawn in straight lines.

A polygon

Then a triangulation of a polygon can be 3-colored.

By induction on $n$, the number of vertices ($= \# \text{ edges}$) in polygon circuit $n = 3$ clear. Let $T$ be a polygon with $n$ edges. Pick a chord edge $e$, e.g., chord $(3, 5)$ in the figure above. Note $T$ must have at least one chord edge, since $n \geq 4$. Chord $e$ splits $T$ into two smaller polygons; by induction each can be triangulated. Color both, choosing colors so edge $e$ has same colors at end vertices (relabeling if needed).
The Art Gallery Problem

- place guards so that all the walls of an art gallery are watched. What is the minimum # of guards needed? Guards need to have a direct line of sight to every point on the walls. Guard at a corner is assumed to be able to see the two walls that end at the corner.

Thus, the Art Gallery Problem with n walls requires at most \( \lceil \frac{n}{3} \rceil \) guards.

Fl. Triangles + color with red, blue, and green.

Place a guard at each red vertex - this guard sees all sides of the triangle, so now each wall is viewed by...
a graph. If there are no cycles, then some color is used at most \( \sqrt{3} \) times.

Example where \( \sqrt{3} \) is needed:

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\[\text{12 edges} \quad \text{Clearly need at least 4 guards}\]
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Then, if a graph \( G \) is not an odd circuit or a complete graph, then \( \chi'(G) \leq d \), where \( d \) is the maximum degree of a vertex of \( G \).

Defn: Edge chromatic \( \chi' = \) minimum \# of colors needed to color edges so edges with common end vertex get different colors.

Thm 4: If max degree of a vertex in \( G \) is \( d \), then the edge chromatic \( \chi'(G) \) is either \( d \) or \( d+1 \).

Thm: Every planar graph can be 5-colored (can actually be 4-colored but proof very long)

Pf: See book page 80-81
For each in proof of Thm. 4. Any planar graph has a vertex of degree \( \leq 5 \).

**Proof:**

Assume not, then \( \sum_{v \in G} \deg(v) \geq 6V \).

But \( \sum_{v \in G} \deg(v) = 2e \) so \( 2e \geq 6V \).

But \( e \leq 3V - 6 \).

So \( 3V - 6 \geq e \geq 3V \) \( \Rightarrow 3V - 6 \geq 3V \) contradiction.

**Chromatic Polynomial**

Define \( P_k(G) \) = \# of ways to color a graph \( G \) with \( K \) colors.

Then \( P_k(G) \) is a polynomial in \( K \).

**Proof:** Given any coloring \( C \) of \( G \), form a set partition \( \Delta(C) \) by placing vertices with the same color in the same block. Conversely, given a set partition \( \Delta \) of \( G \) with the property that two vertices in the
same blocks are not adjacent, one can form a coloring $C(\lambda)$ by coloring vertices in the same block with the same color.

For each such $\lambda$ with $m$ blocks, there are $K(k-1) \cdot (k-m+1)$ ways to color the blocks with $k$ colors. Thus

$$P_k(G) = \sum_{\lambda} K(k-1) \cdot (k-m+1)$$

where $m = \# \text{blocks of } \lambda$.

$$= \sum_{m} K(k-1) \cdot (k-m+1) \sum_{\text{blocks}} 1$$
\[ P_k(G) = k(k-1)(k-2) 5 + k(k-1)(k-2)(k-3) 5 + k(k-1)(k-2)(k-3)(k-4) \]

**Example:** A connected graph with no circuits is called a **tree**. If \( T \) is a tree with \( n \) vertices, then \( P_k(T) = k(k-1)^{n-1} \).