HW 1 - Math 581 - Spring 2020

1. In this problem we work with superwords, which are words in a positive alphabet $\mathcal{A}_+ = \{1, 2, 3, \ldots\}$ and a negative alphabet $\mathcal{A}_- = \{\bar{1}, \bar{2}, \ldots\}$. When standardizing a superword, equal negative letters standardize in the reverse way that equal positive letters do. For example, $\bar{1}122$ standardizes to $4312$ assuming the ordering $1 < 2 < \cdots < n < \bar{n} < \bar{n} - 1 \cdots \bar{1}$ while it standardizes to $2134$ assuming the ordering $1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}$. For a given composition $\alpha$ of $n$, let $S(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, n\}$. Furthermore, let

$$
\tilde{F}_\alpha(X, Y) = \sum_{\substack{a_1 \leq a_2 \leq \cdots \leq a_n \\ a_i = a_{i+1} \in \mathcal{A}_- \implies i \in S(\alpha) \\ a_i = a_{i+1} \in \mathcal{A}_+ \implies i \notin S(\alpha)}} \prod_{a_i \in \mathcal{A}_+} x_{a_i} \prod_{a_i \in \mathcal{A}_-} y_{|a_i|},
$$

be the super Gessel fundamental quasisymmetric function, where $|\bar{i}| = i$. Prove that for any ordering of the alphabets and permutation $\beta \in S_n$,

$$
\tilde{F}_{\text{runs}}(\beta)(X, Y) = \sum_{\text{superwords } \sigma \text{ in } n \text{ letters}} \prod_{a_i \in \mathcal{A}_+} x_{a_{i}} \prod_{a_i \in \mathcal{A}_-} y_{|a_i|}.
$$

2. Let $V = (V_1, V_2, \ldots, V_k)$ be a tuple of skew shapes. Show that Haiman’s sign-reversing involution works for the LLT product of the $V_i$ (where the LLT product is as defined in class and also described on pages 93 – 95, 126 of my book for certain $V$). That is, show the symmetric function $\text{LLT}_V[X(q - 1); q]$ can be expressed as a sum over nonattacking fillings $\tilde{\sigma}$.