10

Combinatorial aspects of Macdonald and related polynomials

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10.1 Introduction

The theory of symmetric functions plays an increasingly important role in modern mathematics, with substantial applications to representation theory, algebraic geometry, special functions, mathematical physics, knot invariants, algebraic combinatorics, statistics, and other areas. We give several references to these applications in the following sections, which involve a family of symmetric functions, Macdonald polynomials, with two extra parameters q, t. This family was introduced by Macdonald in 1988 [120], [119] and contains most of the previously studied important families of symmetric functions, such as Schur functions, Hall-Littlewood polynomials, and Jack polynomials, as limiting or special cases.

Macdonald polynomials can be studied from multiple points of view. Haiman [69] used properties of the Hilbert scheme from algebraic geometry to prove the famous "n! Conjecture" of Garsia and Haiman [37], which says that Macdonald polynomials represent bigraded characters of certain modules of the symmetric group. This implies a Schur-positivity conjecture made by Macdonald in [120].

In 1995 Macdonald [121], by generalizing his construction of 1988, showed that to any affine root system one can associate a family of polynomials which satisfy a multivariate orthogonality condition. This family contains both symmetric polynomials, which are bases for the ring of symmetric functions, and nonsymmetric polynomials, which are bases for the polynomial ring. Further improvements in this direction were made by Cherednik [24]. Combined with earlier work of Koornwinder [87], the resulting construction yields a very general family of polynomials which contains the Askey-Wilson polynomials, and hence all classical orthogonal polynomials, as special cases. See Chapter 9 of this volume for a detailed introduction to this general family.

In the setup of Chapter 9 in this volume, the polynomials introduced by Macdonald in 1988 correspond to a GL_n root system. This chapter focuses on the combinatorics of this case, and also of an associated object, the space of diagonal harmonics DH_n. This space, which has become increasingly important in algebra and representation theory, has a beautiful and remarkably rich combinatorial structure containing two-parameter extensions of Catalan numbers, Schröder numbers, parking functions, and other popular combinatorial objects. Macdonald has commented that the study of root systems and Lie algebras provides an inexhaustible

Much of this chapter is a condensed version of Chapters 1, 2, 6, 7, and Appendix A of the author's book *The q,t-Catalan Numbers and the Space of Diagonal Harmonics: With an Appendix on the Combinatorics of Macdonald Polynomials*, source of wonderful combinatorics. It seems something similar holds for the study of the Hilbert scheme and DH_n.

Section 10.2 contains some background material from the theory of symmetric functions. Section 10.3 contains an overview of the basic analytic and algebraic properties of Macdonald polynomials discovered by Macdonald, Garsia, and Haiman. Section 10.4 covers the combinatorics of the space of diagonal harmonics DH_n, and its connection to Macdonald polynomial theory. In Section 10.5 we discuss the combinatorial formula of Haglund, Haiman, and Loehr giving the expansion of the Macdonald polynomial into monomials. In Section 10.6 we discuss some of the many consequences of this formula. A corresponding combinatorial formula for the expansion of the GL_n nonsymmetric Macdonald polynomial into monomials is the subject of Section 10.7. In Section 10.8 we discuss how results on the combinatorics of DH_n from Section 10.4 led to the monomial formula for Macdonald polynomials in Section 10.5, and Section 10.9 contains a brief overview of other recent approaches to Macdonald polynomials. Section 10.10 overviews some important recent developments which haven't yet been published.

10.2 Basic Theory of Symmetric Functions

This section contains only a brief overview of symmetric function theory; for a more detailed treatment of the subject we refer the reader to [120, Chap. 1] and [142, Chap. 7]. Given $f(x_1, \ldots, x_n) \in K[x_1, x_2, \ldots, x_n]$ for some field K of characteristic 0, and $\sigma \in S_n$, let

$$\sigma f = f(x_{\sigma_1}, \ldots, x_{\sigma_n}). \tag{10.2.1}$$

We say f is a *symmetric function* if $\sigma f = f$, for every σ in the symmetric group S_n. It will be convenient to work with more general functions f depending on countably many indeterminates $x_1, x_2, \ldots$, indicated by $f(x_1, x_2, \ldots)$, in which case we view f as a formal power series in the x_i, and say it is a symmetric function if it is invariant under any permutation of the variables. We let X_n and X stand for the set of variables $\{x_1, \ldots, x_n\}$ and $\{x_1, x_2, \ldots\}$, respectively.

A *partition* λ is a nonincreasing finite sequence $\lambda_1 \geq \lambda_2 \geq \ldots$ of positive integers. λ_i is called the ith *part* of λ. We let $\ell(\lambda)$ denote the number of parts, $|\lambda| = \sum_i \lambda_i$ the sum of the parts, and say λ is a partition of $|\lambda|$. For various formulas it will be convenient to assume $\lambda_j = 0$ for $j > \ell(\lambda)$. The *Ferrers graph* of λ is an array of unit squares, called *cells*, with λ_i cells in the ith row, with the first cell in each row left-justified, and with the first row at the bottom. We often use λ to refer to its Ferrers graph. We define the *conjugate* partition, λ' as the partition whose Ferrers graph is obtained from λ by reflecting across the diagonal $x = y$. See Figure 10.1. By convention $(i, j) \in \lambda$ refers to a cell with (column, row) coordinates (i, j), with the lower left-hand-cell of λ having coordinates $(1, 1)$. The notation $x \in \lambda$ means x is a cell in λ. For technical reasons we say that 0 has one partition, the empty set $\emptyset$, with $\ell(\emptyset) = 0 = |\emptyset|$.

We let Λ^n denote the vector space consisting of symmetric functions in $x_1, x_2, \ldots$ that are homogeneous of degree n. The ring of symmetric functions Λ is the direct sum of the Λ^n. The

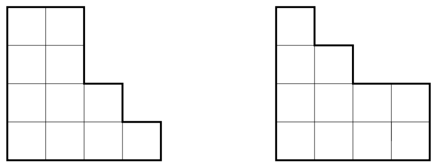

Figure 10.1 On the left, the Ferrers graph of the partition $(4, 3, 2, 2)$, and on the right, that of its conjugate $(4, 3, 2, 2)' = (4, 4, 2, 1)$.

most basic symmetric functions are the *monomial symmetric functions*, which depend on a partition λ in addition to a set of variables. They are denoted by $m_\lambda(X) = m_\lambda(x_1, x_2, \ldots)$. In a symmetric function it is typical to leave off explicit mention of the variables, with a set of variables being understood from context, so $m_\lambda = m_\lambda(X)$. We illustrate these first by means of examples. We let $\mathrm{Par}(n)$ denote the set of partitions of n, and use the notation $\lambda \vdash n$ as an abbreviation for $\lambda \in \mathrm{Par}(n)$. For example,

$$m_{1,1} = \sum_{i<j} x_i x_j, \quad m_{2,1,1}(X_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2, \quad m_2(X) = \sum_i x_i^2.$$

In general, $m_\lambda(X)$ is the sum of all distinct monomials in the x_i whose multiset of exponents equals the multiset of parts of λ. Any element of Λ can be expressed uniquely as a linear combination of the m_λ.

We let 1^n stand for the partition consisting of n parts of size 1. The function m_{1^n} is called the nth *elementary symmetric function*, which we denote by e_n. Then

$$\prod_{i=1}^{\infty}(1 + zx_i) = \sum_{n=0}^{\infty} z^n e_n, \quad e_0 = 1. \tag{10.2.2}$$

Another important special case is $m_n = \sum_i x_i^n$, known as the *power-sum symmetric functions*, denoted by p_n. We also define the *complete homogeneous symmetric function* h_n, by $h_n = \sum_{\lambda \vdash n} m_\lambda$, or equivalently

$$\frac{1}{\prod_{i=1}^{\infty}(1 - zx_i)} = \sum_{n=0}^{\infty} z^n h_n. \tag{10.2.3}$$

For $\lambda \vdash n$, we define $e_\lambda = \prod_i e_{\lambda_i}$, $p_\lambda = \prod_i p_{\lambda_i}$, and $h_\lambda = \prod_i h_{\lambda_i}$. For example,

$$\begin{aligned} e_{2,1} &= \sum_{i<j} x_i x_j \sum_k x_k = m_{2,1} + 3m_{1,1,1} \\ p_{2,1} &= \sum_i x_i^2 \sum_j x_j = m_3 + m_{2,1} \\ h_{2,1} &= \Big(\sum_i x_i^2 + \sum_{i<j} x_i x_j\Big) \sum_k x_k = m_3 + 2m_{2,1} + 3m_{1,1,1}. \end{aligned}$$

It is known that $\{e_\lambda,\ \lambda \vdash n\}$ forms a basis for Λ^n, and so do $\{p_\lambda,\ \lambda \vdash n\}$ and $\{h_\lambda,\ \lambda \vdash n\}$.

Definition 10.2.1 Two simple functions on partitions we will often use are

$$n(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i'}{2}$$
$$z_\lambda = \prod_i i^{n_i} n_i!\,,$$

where $n_i = n_i(\lambda)$ is the number of parts of λ equal to i.

We let ω denote the ring endomorphism $\omega : \Lambda \to \Lambda$ defined by

$$\omega(p_k) = (-1)^{k-1} p_k. \tag{10.2.4}$$

Thus ω is an involution with $\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda$. Also, $\omega(e_n) = h_n$, and more generally $\omega(e_\lambda) = h_\lambda$.

Remark 10.2.2 Identities like $h_{2,1} = m_3 + 2m_{2,1} + 3m_{1,1,1}$ appear at first to depend on a set of variables, but it is customary to view them as polynomial identities in the p_λ. Since the p_k are algebraically independent, we can specialize them to whatever we please, forgetting about the original set of variables X.

We define the Hall scalar product, a bilinear form from $\Lambda \times \Lambda \to \mathbb{Q}$, by

$$\langle p_\lambda, p_\beta \rangle = z_\lambda\, \chi(\lambda = \beta), \tag{10.2.5}$$

where for any logical statement L,

$$\chi(L) = \begin{cases} 1 & \text{if } L \text{ is true} \\ 0 & \text{if } L \text{ is false.} \end{cases} \tag{10.2.6}$$

Clearly $\langle f, g \rangle = \langle g, f \rangle$. Also, $\langle \omega f, \omega g \rangle = \langle f, g \rangle$, which follows from the definition if $f = p_\lambda, g = p_\beta$, and by bilinearity for general f, g since the p_λ form a basis for Λ.

Theorem 10.2.3 (See [120, Chapter I.4] or [142, Chapter 7])
The h_λ and the m_β are dual with respect to the Hall scalar product, i.e.

$$\langle h_\lambda, m_\beta \rangle = \chi(\lambda = \beta). \tag{10.2.7}$$

For any $f \in \Lambda$, and any basis $\{b_\lambda, \lambda \in \mathrm{Par}\}$ of Λ, let $f|_{b_\mu}$ denote the coefficient of b_μ when f is expressed in terms of the $\{b_\lambda\}$. Then (10.2.7) implies

Corollary 10.2.4 $\langle f, h_\lambda \rangle = f|_{m_\lambda}$.

Tableaux and Schur Functions

Given $\lambda, \mu \in \mathrm{Par}(n)$, a *semi-standard Young tableaux* (or SSYT) of shape λ and weight μ is a filling of the cells of the Ferrers graph of λ with the elements of the multiset $\{1^{\mu_1} 2^{\mu_2} \cdots\}$, so that the numbers weakly increase across rows and strictly increase up columns. Let $\mathrm{SSYT}(\lambda, \mu)$ denote the set of these fillings, and $K_{\lambda,\mu}$ the cardinality of this set. The $K_{\lambda,\mu}$ are known as the Kostka numbers. Our definition also makes sense if our weight is a composition of n, i.e. any finite sequence of nonnegative integers whose sum is n. For example, $K_{(3,2),(2,2,1)} = K_{(3,2),(2,1,2)} = K_{(3,2),(1,2,2)} = 2$ as in Figure 10.2.

$$\begin{array}{|c|c|c} \hline 2 & 2 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 2 & 3 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

$$\begin{array}{|c|c|c} \hline 2 & 3 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 3 & 3 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

$$\begin{array}{|c|c|c} \hline 2 & 3 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 3 & 3 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

Figure 10.2 Some SSYT of shape $(3, 2)$.

If the Ferrers graph of a partition β is contained in the Ferrers graph of λ, denoted $\beta \subseteq \lambda$, let λ/β refer to the subset of cells of λ which are not in β. This is referred to as a *skew shape*. Define a SSYT of shape λ/β and weight ν, where $|\nu| = |\lambda| - |\beta|$, to be a filling of the cells of λ/β with elements of $\{1^{\nu_1}2^{\nu_2}\cdots\}$, again with weak increase across rows and strict increase up columns. We let $\mathrm{SSYT}(\lambda, \mu)$ denote the set of such tableaux, and its cardinality by $K_{\lambda/\beta,\nu}$.

Let $\mathrm{wcomp}(\mu)$ denote the set of all compositions whose multiset of nonzero parts equals the multiset of parts of μ. It follows easily from Figure 10.2 that $K_{(3,2),\alpha} = 2$ for all $\alpha \in \mathrm{wcomp}(2, 2, 1)$. Hence

$$\sum_{\alpha,T} \prod_i x_i^{\alpha_i} = 2m_{(2,2,1)}, \tag{10.2.8}$$

where the sum is over all tableaux T of shape $(3, 2)$ and weight some element α of $\mathrm{wcomp}(2, 2, 1)$.

This is a special case of a more general phenomenon. For $\lambda \in \mathrm{Par}(n)$, define

$$s_\lambda = \sum_{\alpha,T} \prod_i x_i^{\alpha_i}, \tag{10.2.9}$$

where the sum is over all compositions α of n, and all possible tableaux T of shape λ and weight α. Then

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu. \tag{10.2.10}$$

The s_λ, called *Schur functions*, are in Λ and are fundamental to the theory of symmetric functions. Two special cases of (10.2.10) are $s_n = h_n$ (since $K_{n,\mu} = 1$ for all $\mu \in \mathrm{Par}(n)$) and $s_{1^n} = e_n$ (since $K_{1^n,\mu} = \chi(\mu = 1^n)$).

A SSYT of weight 1^n is called *standard*, or a SYT. The set of SYT of shape λ is denoted $\mathrm{SYT}(\lambda)$. Below we list some of the important properties of Schur functions.

Theorem 10.2.5 *Let* $\lambda, \mu \in \mathrm{Par}$. *Then*

1. *The Schur functions are orthonormal with respect to the Hall scalar product, i.e.,* $\langle s_\lambda, s_\mu \rangle = \chi(\lambda = \mu)$. *Thus, for any* $f \in \Lambda$, $\langle f, s_\lambda \rangle = f|_{s_\lambda}$.
2. *Action by* ω*:* $\omega(s_\lambda) = s_{\lambda'}$.
3. (The Jacobi-Trudi identity)
$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1}^{\ell(\lambda)}$, *where we set* $h_0 = 1$ *and* $h_k = 0$ *for* $k < 0$.
4. (The Pieri rule) *Let* $k \in \mathbb{N}$. *Then*

$$s_\lambda h_k = \sum_\gamma s_\gamma, \tag{10.2.11}$$

where the sum is over all γ *whose Ferrers graph contains* λ *with* $|\gamma/\lambda| = k$ *and such that* γ/λ *is a "horizontal strip", i.e., has no two cells in the same column. Note that by applying* ω *to both sides of* (10.2.11) *we can get a corresponding expression for* $s_\lambda e_k$. *For example,*

$$s_{2,1}h_2 = s_{4,1} + s_{3,2} + s_{3,1,1} + s_{2,2,1}, \quad s_{2,1}e_2 = s_{2,1,1,1} + s_{2,2,1} + s_{3,1,1} + s_{3,2}.$$

Statistics on Tableaux

There is a q-analogue of the Kostka numbers, denoted by $K_{\lambda,\mu}(q)$, which has many applications. They can be defined as the coefficients that arise when expanding Schur functions into Hall-Littlewood polynomials [120, Chapter II.6]. The $K_{\lambda,\mu}(q)$ are polynomials in q which satisfy $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$. Foulkes [31] conjectured that there should be a statistic stat(T) on $T \in \mathrm{SSYT}(\lambda, \mu)$ such that

$$K_{\lambda,\mu}(q) = \sum_{T \in SSYT(\lambda,\mu)} q^{\mathrm{stat}(T)}. \tag{10.2.12}$$

This conjecture was resolved by Lascoux and Schützenberger [101], who found a statistic *charge* to generate these polynomials. Butler [21] provided a detailed account of their proof, filling in a lot of missing details. A short proof, based on the new combinatorial formula for Macdonald polynomials, is contained in Section 10.6.

Let $T \in \mathrm{SSYT}(\lambda, \mu)$, and let cocharge$(T) = n(\mu) -$ charge(T). The *reading word* read(T) of T is obtained by reading the entries in T from left to right in the top row of T, then continuing left to right in the second row from the top of T, etc. For example, the tableau in the upper-left of Figure 10.2 has reading word 22113. To calculate cocharge(T), perform the following algorithm on read(T).

Cocharge Algorithm

1. Start at the end of read(T) and scan left until you encounter a 1 - say this occurs at spot i_1, so read$(T)_{i_1} = 1$. Then start there and scan left until you encounter a 2. If you hit the end of read(T) before finding a 2, loop around and continue searching left, starting at the end of read(T). Say the first 2 you find equals read$(T)_{i_2}$. Now iterate, start at i_2 and search left until you find a 3, etc. Continue in this way until you have found $4, 5, \ldots, \mu'_1$, with μ'_1 occurring at spot $i_{\mu'_1}$. Then the first subword of read(T) is defined to be the elements of the set

$\{\mathrm{read}(T)_{i_1}, \ldots, \mathrm{read}(T)_{i_{\mu'_1}}\}$, listed in the order in which they occur in read(T) if we start at the beginning of read(T) and move left to right. For example, if read(T) = 21613244153 is a word of content $\mu = (3, 2, 2, 2, 1, 1)$, then the first subword equals 632415, corresponding to places $3, 5, 6, 8, 9, 10$ of read(T).

Next remove the elements of the first subword from read(T) and find the first subword of what's left. Call this the second subword of read(T). Remove this and find the first subword in what's left and call this the third subword of read(T), etc. For the word 21613244153, the subwords are 632415, 2143, 1.

2. The value of cocharge(T) will be the sum of the values of cocharge on each of the subwords of read(T). Thus it suffices to assume read(T) $\in S_m$ for some m, in which case we set

$$\mathrm{cocharge}(\mathrm{read}(T)) = \mathrm{comaj}(\mathrm{read}(T)^{-1}),$$

where $\mathrm{read}(T)^{-1}$ is the usual inverse in S_m. Here comaj(σ) is equal to the sum of $m - i$ over those i in the descent set Des(σ), i.e., over those i for which $\sigma_i > \sigma_{i+1}$. For example, if $\sigma = 632415$, then $\sigma^{-1} = 532461$, cocharge(σ) $= 5 + 4 + 1 = 10$, and so

$$\mathrm{cocharge}(21613244153) = 10 + 4 + 0 = 14.$$

Note that to compute charge, we could create subwords in the same manner, and count $m-i$ for each i with $i + 1$ occurring to the right of i instead of to the left. Set

$$\tilde{K}_{\lambda,\mu}(q) = q^{n(\mu)} K_{\lambda,\mu}(1/q) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} q^{\mathrm{cocharge}(T)}. \tag{10.2.13}$$

These polynomials have various interpretations in terms of representation theory and geometry [40], [118], [75].

In addition to the cocharge statistic, there is a major index statistic on SYT which is often useful. Given a SYT tableau T of shape λ , define a descent of T to be a value of i, $1 \le i < |\lambda|$, for which $i + 1$ occurs in a row above i in T. Let

$$\mathrm{maj}(T) = \sum i, \qquad \mathrm{comaj}(T) = \sum |\lambda| - i, \tag{10.2.14}$$

where the sums are over the descents of T. Then [142, p.363]

$$s_\lambda(1, q, q^2, \ldots,) = \frac{1}{(q;q)_n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \frac{1}{(q;q)_n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{comaj}(T)}, \tag{10.2.15}$$

where $(w; q)_n = (1 - w)(1 - wq) \cdots (1 - wq^{n-1})$ is the usual q-rising factorial.

Plethystic Notation

Many of the theorems later in this chapter involving symmetric functions will be expressed in plethystic notation. In this subsection we define this and give several examples in order to acclimate the reader. For more detailed treatments of plethysm see [42], [43], [120, I.8].

Let $E(t_1, t_2, t_3 \ldots)$ be a formal series of rational functions in the parameters $t_1, t_2, \ldots$. We define the plethystic substitution of E into p_k, denoted $p_k[E]$, by

$$p_k[E] = E(t_1^k, t_2^k, \ldots). \tag{10.2.16}$$

Note the square "plethystic" brackets around E - this is to distinguish $p_k[E]$ from the ordinary kth power sum in a set of variables E, which we have already defined as $p_k(E)$. One thing we need to emphasize is that any minus signs occurring in the definition of E are left as is when replacing the t_i by t_i^k.

Example 10.2.6

1. Inside plethystic brackets, we view a set of variables X as $p_1(X) = x_1 + x_2 + \ldots$. For example, $p_k[X] = p_k(X)$.
2. For z an indeterminate, $p_k[zX] = z^k p_k[X]$.
3. $p_k[X - Y] = \sum_i (x_i^k - y_i^k) = p_k[X] - p_k[Y]$.
4. $p_k\left[\dfrac{X(1-z)}{1-q}\right] = \displaystyle\sum_i \frac{x_i^k(1-z^k)}{1-q^k}$.
5. $\displaystyle\prod_i \frac{(1 - tx_i z)}{(1 - x_i)z} = \sum_{n=0}^{\infty} z^n h_n[X(1-t)]$,

   which can be proved by taking exp ln of the left-hand-side, and expressing everything in terms of the p_k.

Let $Z = (-x_1, -x_2, \ldots)$. Note that $p_k(Z) = \sum_i (-1)^k x_i^k$, which is different from $p_k[-X]$. We need a special notation for the case where we wish to replace variables by their negatives inside plethystic brackets. We use the ϵ symbol to denote this, i.e.

$$p_k[\epsilon X] = \sum_i (-1)^k x_i^k. \tag{10.2.17}$$

We now extend this definition of plethystic substitution of E into f for an arbitrary $f \in \Lambda$ by first expressing f as a polynomial in the p_k, say $f = \sum_\lambda c_\lambda p_\lambda$ for constants c_λ, then defining $f[E]$ as

$$f[E] = \sum_\lambda c_\lambda \prod_i p_{\lambda_i}[E]. \tag{10.2.18}$$

We mention that for any $f \in \Lambda$, $\omega(f(X)) = f[-\epsilon X]$.

Some particularly useful plethystic identities are the following "addition formulas". They can be proved by first expressing them in terms of the p_λ (see [53, Chap. 1]). The same identities hold if replace h_k by e_k throughout.

Theorem 10.2.7 *Let $E = E(t_1, t_2, \ldots)$ and $F = F(w_1, w_2, \ldots)$ be two formal series of rational terms in their indeterminates. Then*

$$h_n[E + F] = \sum_{k=0}^{n} h_k[E] h_{n-k}[F], \quad h_n[E - F] = \sum_{k=0}^{n} h_k[E] h_{n-k}[-F]. \tag{10.2.19}$$

The Fundamental Basis for the Ring of Quasisymmetric Functions

A multivariate polynomial $f(X)$ is called *quasisymmetric* if the coefficient of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ in f is equal to the coefficient of $x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$ in f whenever $1 \le i_1 < i_2 < \cdots < i_k$ and $1 \le j_1 < j_2 < \cdots < j_k$, for all $a_1, a_2, \ldots, a_k \in \mathbb{N}$. For a subset S of $\{1, 2, \ldots, n-1\}$, let

$$F_{n,S}(X) = \sum_{\substack{1 \le a_1 \le a_2 \le \cdots \le a_n \\ a_i = a_{i+1} \implies i \notin S}} x_{a_1} x_{a_2} \cdots x_{a_n} \tag{10.2.20}$$

denote Gessel's fundamental quasisymmetric function. The $F_{n,S}$ form an important basis for the ring of quasisymmetric functions.

Remark 10.2.8 There is another way to view $F_{n,S}(X)$ which will prove useful later. For any word $b_1 b_2 \cdots b_n$ of positive integers, let the *standardization* $\zeta(b_1 b_2 \cdots b_n)$ denote the permutation in S_n which satisfies $\zeta_i < \zeta_j$ if and only if $b_i \le b_j$, for all $1 \le i < j \le n$. For example, $\zeta(23253) = 13254$. Then for any $\sigma \in S_n$, $F_{n,\mathrm{Des}(\sigma^{-1})}(X)$ is simply the sum of $x_{b_1} x_{b_2} \cdots x_{b_n}$, over all words $b_1 b_2 \cdots b_n$ of positive integers, whose standardization $\zeta(b_1 b_2 \cdots b_n)$ equals σ.

Graded Hilbert Series and Characters

This section assumes some basic facts about the representation theory of finite groups which will be familiar to many readers. Good sources for background information on these topics are [79] and [135]. Let G be a finite group, and V a finite dimensional $\mathbb{C}$ vector space, with basis $w_1, w_2, \ldots w_n$. Any linear action of G on V makes V into a $\mathbb{C}G$ module. A module is called *irreducible* if it has no submodules other than $\{0\}$ and itself. Every finite dimensional $\mathbb{C}G$-module V can be expressed as a direct sum of irreducible submodules.

The character of the module (under the given action), which we denote $\mathrm{char}(V)$, is a function on G depending only on the conjugacy class of its argument. The character is called irreducible if V is irreducible. The irreducible characters form a basis of the space of conjugation invariant functions on G, so the number of conjugacy classes is equal to the number of irreducible characters. If $V = \bigoplus_{j=1}^{d} V_j$, where each V_j is irreducible, then $\mathrm{char}(V) = \sum_{j=1}^{d} \mathrm{char}(V_j)$.

For the symmetric group S_n the conjugacy class of an element σ is determined by rearranging the lengths of the disjoint cycles of σ into nonincreasing order to form a partition, called the cycle-type β of σ. See [135] The possible cycle-types are precisely the partitions of n. The number of elements in the conjugacy class determined by β is equal to $n!/z_\beta$. For a character χ and an element σ with cycle-type β we can write $\chi(\beta)$ instead of $\chi(\sigma)$. Moreover there is a canonical bijection $\lambda \mapsto \chi^\lambda$ of $\mathrm{Par}(n)$ onto the set of irreducible characters. This bijection can for instance be given by the equivalent identities in Theorem 10.2.9 below.

The dimension of the irreducible S_n-module with character χ^λ is known to be f^λ, the number of SYT of shape λ. For example, in dimension 1 we have two characters, $\chi^{(n)}$ (called the trivial character since $\chi^{(n)}(\mu) = 1$ for all $\mu \vdash n$) and χ^{1^n} (called the sign character since $\chi^{1^n}(\mu) = (-1)^{n-\ell(\mu)}$ which is the sign of any permutation of cycle type μ). One reason Schur functions are important in the representation theory of S_n is the following.

Theorem 10.2.9 *When expanding the p_μ into the s_λ basis, the coefficients are the χ^λ. To be exact:*

$$p_\mu = \sum_{\lambda \vdash n} \chi^\lambda(\mu) s_\lambda, \quad s_\lambda = \sum_{\mu \vdash n} z_\mu^{-1} \chi^\lambda(\mu) p_\mu.$$

Let V be a graded subspace of $\mathbb{C}[x_1, \ldots, x_n]$, with respect to the grading of $\mathbb{C}[x_1, \ldots, x_n]$ by degree of homogeneity. Then $V = \bigoplus_{i=0}^{\infty} V^{(i)}$, where $V^{(i)}$ is the finite dimensional subspace consisting of all elements of V that are homogeneous of degree i in the x_j. We define the *Hilbert series* $\mathrm{Hilb}(V; q)$ of V to be the sum

$$\mathrm{Hilb}(V; q) = \sum_{i=0}^{\infty} q^i \dim(V^{(i)}), \tag{10.2.21}$$

where dim indicates the dimension as a $\mathbb{C}$ vector space.

Given $f(x_1, \ldots, x_n) \in \mathbb{C}[X_n]$ and $\sigma \in S_n$, setting $\sigma f = f(x_{\sigma_1}, \ldots, x_{\sigma_n})$ defines an action of S_n on $\mathbb{C}[X_n]$. Assume that V is as above, is fixed by the S_n action, which also respects the grading. We define the *Frobenius series* $\mathrm{Frob}(V; X, q)$ of V to be the symmetric function

$$\mathrm{Frob}(V; X, q) := \sum_{i=0}^{\infty} q^i \sum_{\lambda \in \mathrm{Par}(i)} \mathrm{Mult}\,(\chi^\lambda, V^{(i)}) s_\lambda(X), \tag{10.2.22}$$

where $\mathrm{Mult}\,(\chi^\lambda, V^{(i)})$ is the multiplicity of the irreducible character χ^λ in the character of $V^{(i)}$ under the action. In other words, if we decompose $V^{(i)}$ into irreducible S_n-submodules, $\mathrm{Mult}\,(\chi^\lambda, V^{(i)})$ is the number of these submodules whose trace equals χ^λ. We will typically refer to the Frobenius series by the simpler notation $\mathrm{Frob}(V; q)$, leaving out the reference to the implicit set of variables on both sides of (10.2.22).

A polynomial in $\mathbb{C}[X_n]$ is *alternating*, or an alternate, if

$$\sigma f = (-1)^{\mathrm{inv}(\sigma)} f \quad \forall\, \sigma \in S_n, \tag{10.2.23}$$

where $\mathrm{inv}(\sigma)$ is the number of inversions of σ, i.e the number of pairs (i, j) with $1 \le i < j \le n$ and $\sigma_i > \sigma_j$. The set of alternates in V forms a subspace called the subspace of alternates, or anti-symmetric elements, denoted V^ϵ. This is also an S_n-submodule of V.

Remark 10.2.10 Since the dimension of the representation corresponding to χ^λ equals f^λ, which also equals the coefficient of m_{1^n} in s_λ, by Corollary (10.2.4) we have

$$\langle \mathrm{Frob}(V; q), h_{1^n} \rangle = \mathrm{Hilb}(V; q). \tag{10.2.24}$$

Also, it is an easy exercise to show

$$\langle \mathrm{Frob}(V; q), s_{1^n} \rangle = \mathrm{Hilb}(V^\epsilon; q). \tag{10.2.25}$$

Example 10.2.11 Since $\dim(\mathbb{C}[X_n]^{(i)}) = \binom{n+i-1}{i}$, $\mathrm{Hilb}(\mathbb{C}[X_n]; q) = (1-q)^{-n}$. Taking into account the S_n-action, it is known that [11, Sec. 8.5]

$$\mathrm{Frob}(\mathbb{C}[X_n]; q) = \sum_{\lambda \in \mathrm{Par}(n)} s_\lambda \frac{\sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)}}{(q)_n}. \tag{10.2.26}$$

The Ring of Coinvariants

The set of symmetric polynomials in the x_i, denoted $\mathbb{C}[X_n]^{S_n}$, which is generated by $1, e_1, \dots e_n$, is called the *ring of invariants*. Although we will focus on the type A case, we refer the reader to the excellent book [76] by Humphreys for general information on how many of these results apply to more general reflection groups. The quotient ring $\mathrm{R}_n = \mathbb{C}[x_1, \dots, x_n]/\langle e_1, e_2, \dots, e_n\rangle$, or equivalently $\mathbb{C}[x_1, \dots, x_n]/\langle p_1, p_2, \dots, p_n\rangle$, obtained by moding out by the ideal generated by all symmetric polynomials of positive degree is known as the *ring of coinvariants*. It is known that $\dim(\mathrm{R}_n) = n!$ as a $\mathbb{C}$-vector space, and moreover that $\mathrm{Hilb}(\mathrm{R}_n; q) = [n]!$, where $[n]! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$. E. Artin [7] derived a specific basis for R_n, namely cosets of the elements in the set

$$\{\prod_{1\le i\le n} x_i^{\alpha_i},\ 0 \le \alpha_i \le i-1\}. \tag{10.2.27}$$

Also, [141], [143]

$$\mathrm{Frob}(\mathrm{R}_n; q) = \sum_{\lambda\in\mathrm{Par}(n)} s_\lambda \sum_{T\in SYT(\lambda)} q^{\mathrm{maj}(T)}. \tag{10.2.28}$$

Let

$$\Delta = \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ & & \vdots & \\ 1 & x_n & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n} (x_j - x_i)$$

be the Vandermonde determinant. The *space of harmonics* H_n can be defined as the $\mathbb{C}$ vector space spanned by Δ and its partial derivatives of all orders. Haiman [67] provides a detailed proof that H_n is isomorphic to R_n as an S_n module, and notes that an explicit isomorphism α is obtained by letting $\alpha(h), h \in \mathrm{H}_n$, be the element of R_n represented modulo $< e_1, \dots, e_n >$ by h. Thus $\dim(\mathrm{H}_n) = n!$ and moreover the character of H_n under the S_n-action is given by (10.2.28). He also argues that (10.2.28) follows immediately from (10.2.26) and the fact that H_n generates $\mathbb{C}[X_n]$ as a free module over $\mathbb{C}[X_n]^{S_n}$.

10.3 Analytic and Algebraic Properties of Macdonald Polynomials

Macdonald's Original Construction

During the 1980's a number of extensions of Selberg's integral were found. (See Chapter 11 in this volume for background on Selberg's integral.) Askey [8] obtained a q-analogue of the integral, while other generalizations involved the insertion of symmetric functions in the x_i into the integrand (see for example [145]). One of these extensions, due to Kadell [80], involved inserting symmetric functions depending on a partition, a set of variables X_n and another parameter. They are now known as Jack symmetric functions since they were first studied by H. Jack [78].

In his article Kadell gave evidence that a q-analogue of the Jack symmetric functions existed which featured in a q-analogue of his extension of Selberg's integral. Shortly after he proved these polynomials existed for $n = 2$ [81]. The case for general n was solved by Macdonald [119], and these q-analogues of Jack symmetric functions are now called Macdonald polynomials, denoted $P_\lambda(X; q, t)$. A brief discussion of their connection to Kadell's work can also be found in [120, p.387]. The $P_\lambda(X; q, t)$ are symmetric functions with coefficients in $\mathbb{Q}(q, t)$. If we let $q = t^\alpha$, divide by $(1-t)^{|\lambda|}$ and let $t \to 1^-$ in the P_λ we get the Jack symmetric functions with parameter α. Many other important bases of the ring of symmetric functions are also limiting or special cases of the $P_\lambda(X; q, t)$, and their introduction was a major breakthrough in algebraic combinatorics and special functions. In particular, for any q we have $P_\lambda(X; q, q) = s_\lambda(X)$, for any q we have $P_\lambda(X; q, 1) = m_\lambda(X)$, we have $P_\lambda(X; 0, t) = P_\lambda(X; t)$ (the Hall-Littlewood polynomial), and we have $P_\lambda(X; 1, t) = \prod_i e_{\lambda'_i}$. Macdonald polynomials have found applications to many areas including algebraic geometry, mathematical physics, and representation theory [69], [70], [46], [22], [55], [48], [71], [72].

Here is Macdonald's construction of the $P_\lambda(X; q, t)$. The best reference for their basic properties is [120, Chap. 6]. The definition involves the following standard partial order on partitions $\lambda, \mu \in \mathrm{Par}(n)$, called *dominance order*.

$$\lambda \geq \mu \iff \sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j \quad \text{for } i \geq 1. \tag{10.3.1}$$

Theorem 10.3.1 *Define a (q, t)-extension of the Hall scalar product by*

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \chi(\lambda = \mu)\, z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}. \tag{10.3.2}$$

Then the following conditions uniquely define a family of symmetric functions $\{P_\lambda(X; q, t), \lambda \in \mathrm{Par}(n)\}$ *with coefficients in* $\mathbb{Q}(q, t)$:

$$(i) \quad P_\lambda = \sum_{\mu \leq \lambda} c_{\lambda,\mu} m_\mu, \quad \textit{where} \quad c_{\lambda,\mu} \in \mathbb{Q}(q, t) \quad \textit{and} \quad c_{\lambda,\lambda} = 1; \tag{10.3.3}$$

$$(ii) \quad \langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \textit{if } \lambda \neq \mu. \tag{10.3.4}$$

Remark 10.3.2 Since the (q, t)-extension of the Hall scalar product reduces to the ordinary Hall scalar product when $q = t$, it is clear that $P_\lambda(X; q, q) = s_\lambda(X)$. We also note that, since the dominance partial order is not a total order, it is not at all obvious that conditions (10.3.3) and (10.3.4) define a *unique* set of polynomials. Indeed, as Macdonald explains in [120], given any extension of the dominance partial order to a total order, we can apply Gram-Schmidt orthogonalization to obtain a family of symmetric functions satisfying (10.3.3) and (10.3.4). His theorem says that we get the same family no matter which extension to a total order we use. Macdonald proves Theorem 10.3.1 by first constructing operators for which the P_λ are simultaneous eigenfunctions with distinct eigenvalues.

Example 10.3.3 For $\mu \vdash n$, $P_\mu(X; 0, t)$ is known as the *Hall-Littlewood polynomial*, denoted

$P_\mu(X;t)$. Its integral form $Q_\mu(X;t)$ is defined as

$$Q_\mu(X;t) = \prod_i (t)_{n_i(\lambda)} P_\mu(X;t). \tag{10.3.5}$$

If we expand the $Q_\mu(X;t)$ in terms of the Schur basis, the coefficients will be polynomials in $\mathbb{Z}[t]$, but not generally in $\mathbb{N}[t]$, i.e. will not have positive coefficients. Lascoux and Schützenberger proved though that

$$Q_\mu(X;t) = \sum_\lambda s_\lambda[X(1-t)] K_{\lambda,\mu}(t), \tag{10.3.6}$$

where $K_{\lambda,\mu}(t)$ is as in (10.2.13). Note: Using item 5 from Example 10.2.6 one can show that $s_\lambda[X(1-t)]$ is the same as the $S_\lambda(X;t)$ occurring in [120, (III.4.6)].

Given a cell $x \in \lambda$, let the arm $a = a(x)$, leg $l = l(x)$, coarm $a' = a'(x)$, and coleg $l' = l'(x)$ be the number of cells strictly between x and the border of λ in the E, N, W and S directions, respectively, as in Figure 10.3. Also, define

$$B_\mu = B_\mu(q,t) = \sum_{x\in\mu} q^{a'} t^{l'}, \quad \Pi_\mu = \Pi_\mu(q,t) = \prod_{x\in\mu}{}' (1 - q^{a'} t^{l'}), \tag{10.3.7}$$

where a prime symbol $'$ above a product or a sum over cells of a partition μ indicates we ignore the corner $(1,1)$ cell, and $B_\emptyset = 0$, $\Pi_\emptyset = 1$. For example, $B_{(2,2,1)} = 1 + q + t + qt + t^2$ and $\Pi_{(2,2,1)} = (1-q)(1-t)(1-qt)(1-t^2)$. Note that

$$n(\mu) = \sum_{x\in\mu} l' = \sum_{x\in\mu} l.$$

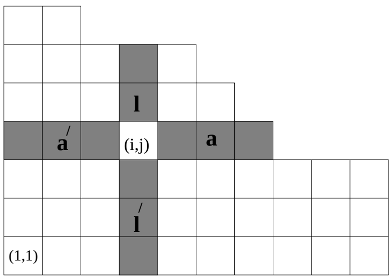

Figure 10.3 The arm a, coarm a', leg l and coleg l' of a cell.

Here are some basic results of Macdonald on the P_λ. A generalization of Property (2) will soon be particularly useful to us. Recall that for any symmetric function F, square brackets as in $F[Z]$ indicate plethystic substitution, and if $\{t_1, t_2, \ldots\}$ is a set of positive parameters, $F[t_1 + t_2 + \ldots] = F(t_1, t_2, \ldots)$.

Theorem 10.3.4 *Let* $\lambda, \mu \in$ Par.

1. *Let* z *be an indeterminate. Then* [120, (VI.6.17)]
$$P_\lambda\left[\frac{1-z}{1-t}; q, t\right] = \prod_{x\in\lambda} \frac{t^{l'} - q^{a'} z}{1 - q^a t^{l+1}}. \tag{10.3.8}$$
2. (Koornwinder-Macdonald Reciprocity)
Assume $n \geq \max(\ell(\lambda), \ell(\mu))$. *Then* [120, (VI.6.6),(VI.6.11)]
$$\frac{\prod_{x\in\mu}(1 - q^a t^{l+1}) P_\mu\left[\sum_{i=1}^n t^{n-i} q^{\lambda_i}; q, t\right]}{\prod_{x\in\mu}(t^{l'} - q^{a'} t^n)} \tag{10.3.9}$$
is symmetric in μ, λ, *where as usual we let* $\mu_i = 0$ *for* $i > \ell(\mu)$, $\lambda_i = 0$ *for* $i > \ell(\lambda)$.
3. *For any two sets of variables* X, Y *(see* [42, Sec. 1]*),*
$$h_n\left[XY\frac{1-t}{1-q}\right] = \sum_{\lambda\vdash n} \frac{\prod_{x\in\lambda}(1 - q^a t^{l+1})}{\prod_{x\in\lambda}(1 - q^{a+1} t^l)} P_\lambda(X; q, t) P_\lambda(Y; q, t), \tag{10.3.10}$$
$$e_n[XY] = \sum_{\lambda\vdash n} P_\lambda(X; q, t) P_{\lambda'}(Y; t, q). \tag{10.3.11}$$
(Identity (10.3.10) *follows from* [120, Chapter VI, (4.13) and (6.19)]
$$\prod_{i,j} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_\lambda \frac{\prod_{x\in\lambda}(1 - q^a t^{l+1})}{\prod_{x\in\lambda}(1 - q^{a+1} t^l)} P_\lambda(X; q, t) P_\lambda(Y; q, t) \tag{10.3.12}$$
by taking the portion of both sides of (10.3.12) *of homogeneous degree* n *in the* X *and* Y *variables.)*

Remark 10.3.5 Let $\lambda \vdash n$, and z an indeterminate. Then $s_\lambda[1 - z] = 0$ if λ is not a "hook" (a hook shape is where $\lambda_2 \leq 1$), in fact
$$s_\lambda[1 - z] = \begin{cases} (-z)^r (1 - z) & \text{if } \lambda = (n - r, 1^r), \quad 0 \leq r \leq n - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{10.3.13}$$
This follows by setting $q = t = 0$ in (10.3.8), since $P_\lambda(X; 0, 0) = s_\lambda$.

The **q, t**-*Kostka Polynomials*

Macdonald found that the $P_\lambda(X; q, t)$ have a very mysterious property. Let $J_\mu[X; q, t]$ denote the so-called Macdonald integral form, defined as
$$J_\mu(X; q, t) = \prod_{x\in\mu} (1 - q^a t^{l+1}) P_\mu(X; q, t). \tag{10.3.14}$$
Now expand J_μ in terms of the $s_\lambda[X(1 - t)]$:
$$J_\mu(X; q, t) = \sum_{\lambda\vdash|\mu|} K_{\lambda,\mu}(q, t) s_\lambda[X(1 - t)] \tag{10.3.15}$$

for some $K_{\lambda,\mu}(q,t) \in \mathbb{Q}(q,t)$. Macdonald conjectured that $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$. This became a famous problem in combinatorics known as Macdonald's positivity conjecture.

Part of the fascination for this conjecture is the case $q = 0$, since $J_\mu(X;0,t) = Q_\mu(X;t)$, and so by (10.3.6) we have

$$K_{\lambda,\mu}(0,t) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{charge}(T)}. \tag{10.3.16}$$

No two-parameter generalization of charge that generates the $K_{\lambda,\mu}(q,t)$ has ever been found though. Macdonald was also able to show that

$$K_{\lambda,\mu}(1,1) = K_{\lambda,\mu}. \tag{10.3.17}$$

The $K_{\lambda,\mu}(q,t)$ are known as the *q,t-Kostka polynomials*.

In the next section we describe a conjecture of Garsia and Haiman which gave a representation-theoretic interpretation for the positivity of the $K_{\lambda,\mu}(q,t)$. This conjecture was proved by Haiman in 2001, resolving Macdonald's positivity conjecture after more than ten years of intensive research.

Macdonald posed a refinement of his positivity conjecture which is still open. Due to (10.3.16) and (10.3.17), one could hope to find statistics $\mathrm{qstat}(T,\mu)$ and $\mathrm{tstat}(T,\mu)$ given by some combinatorial rule so that

$$K_{\lambda,\mu}(q,t) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{qstat}(T,\mu)} t^{\mathrm{tstat}(T,\mu)}. \tag{10.3.18}$$

In Garsia and Haiman's work it is more natural to deal with the polynomials

$$\tilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)} K_{\lambda,\mu}(q,1/t), \tag{10.3.19}$$

so

$$\tilde{K}_{\lambda,\mu}(0,t) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} q^{\mathrm{cocharge}(T)}. \tag{10.3.20}$$

Macdonald found a statistical description of the $K_{\lambda,\mu}(q,t)$ whenever $\lambda = (n-k,1^k)$ is a hook shape [120, Ex. 2, p. 362], which can be stated as

$$\tilde{K}_{(n-k,1^k),\mu} = e_k[B_\mu - 1]. \tag{10.3.21}$$

For example, $\tilde{K}_{(3,1,1),(2,2,1)}(q,t) = e_2[q+t+qt+t^2] = qt + q^2t + 2qt^2 + t^3 + qt^3$. He also found a statistical description when q is set equal to 1 [120, Ex. 7, p. 365], and a similar description when $t = 1$. To describe it, say we are given a statistic $\mathrm{stat}(T)$ on skew SYT, a SYT T with n cells, and a composition $\alpha = (\alpha_1,\ldots,\alpha_k)$ of n into k parts. Define the α-sectionalization of T to be the set of k skew SYT obtained in the following way. The first element of the set is the portion of T containing the numbers 1 through α_1. The second element of the set is the portion of T containing the numbers α_1+1 through $\alpha_1+\alpha_2$, but with α_1 subtracted from each of these numbers, so we end up with a skew SYT of size α_2. In general, the ith element of the set, denoted $T^{(i)}$, is the portion of T containing the numbers $\alpha_1+\ldots+\alpha_{i-1}+1$ through

$\alpha_1 + \ldots + \alpha_i$, but with $\alpha_1 + \ldots + \alpha_{i-1}$ subtracted from each of these numbers. Then we define the α-sectionalization of stat(T), denoted stat(T, α), to be the sum

$$\mathrm{stat}(T, \alpha) = \sum_{i=1}^{k} \mathrm{stat}(T^{(i)}). \tag{10.3.22}$$

In the above terminology, Macdonald's formula for the $q = 1$ Kostka numbers can be expressed as

$$\tilde{K}_{\lambda,\mu}(1, t) = \sum_{T \in \mathrm{SYT}(\lambda)} t^{\mathrm{comaj}(T, \mu')}. \tag{10.3.23}$$

For example, given the tableau T in Figure 10.4 with $\lambda = (4, 3, 2)$ and (coincidentally) μ also $(4, 3, 2)$, then $\mu' = (3, 3, 2, 1)$ and the values of comaj(T, μ') on $T^{(1)}, \ldots, T^{(4)}$ are $1, 2, 1, 0$, respectively, so comaj(T, μ') = 4.

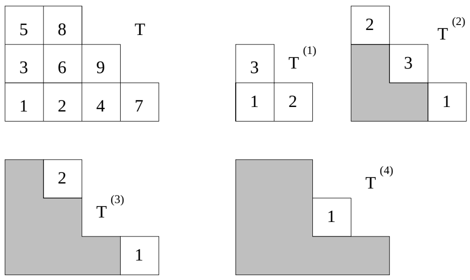

Figure 10.4 The (3, 3, 2, 1)-sections of a SYT.

A combinatorial description of the $\tilde{K}_{\lambda,\mu}(q, t)$ when μ is a hook was found by Stembridge [146]. Given a composition α into k parts, define $\mathrm{rev}(\alpha) = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$. Then if $\mu = (n - k, 1^k)$, Stembridge's result can be expressed as

$$\tilde{K}_{\lambda,\mu}(q, t) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T, \mu)} t^{\mathrm{comaj}(T, \mathrm{rev}(\mu'))}. \tag{10.3.24}$$

Macdonald obtained two symmetry relations, which (expressed in terms of the $\tilde{K}_{\lambda,\mu}$) are

$$\tilde{K}_{\lambda,\mu}(q, t) = \tilde{K}_{\lambda,\mu'}(t, q) \tag{10.3.25}$$

$$\tilde{K}_{\lambda',\mu}(q, t) = t^{n(\mu)} q^{n(\mu')} \tilde{K}_{\lambda,\mu}(1/q, 1/t). \tag{10.3.26}$$

Fischel [29] first obtained statistics for the case when μ has two columns. Using (10.3.26) this also implies statistics for the case where μ has two rows. Later Lapointe and Morse [93] and Zabrocki [150] independently found alternate descriptions of this case, but all of these are rather complicated to state. A simpler description of the two-column case, based on the combinatorial formula for Macdonald polynomials in Section 10.5, is contained in Section 10.6.

In 1996 several groups of researchers [41], [42], [83] ,[84], [85], [98], [99], [136], between them using at least three totally different approaches, independently proved that $\tilde{K}_{\lambda,\mu}(q,t)$ is a polynomial with integer coefficients, which itself had been a major unsolved problem since 1988. (The first breakthrough on this problem appears to have been work of Lapointe and Vinet [96], [97] in 1995, who proved the corresponding integrality result for Jack polynomials. This seemed to have the effect of breaking the ice, since it was shortly after this that the proofs of Macdonald integrality were announced. As in the work of Kadell on Selberg's integral [80], [81], this gives another example of how results in Macdonald theory are often preceded by results on Jack polynomials.) We should mention that the Macdonald polynomiality result is immediately implied by the combinatorial formula in Section 10.5. The paper by Garsia and Remmel [41] also contains a recursive formula for the $\tilde{K}_{\lambda,\mu}(q,t)$ when λ is an augmented hook, i.e. a hook plus the square $(2,2)$. Their formula immediately implies nonnegativity and by iteration could be used to obtain various combinatorial descriptions for this case. In the late 1990's Tesler announced that using plethystic methods he could prove nonnegativity of the case where λ is a "doubly augmented hook", which is an augmented hook plus either the cell $(2,3)$ or $(3,2)$ [147].

The Garsia-Haiman Modules and the $\mathbf{n!}$*-Conjecture*

Given any bihomogeneous subspace $W \subseteq \mathbb{C}[X_n, Y_n]$, we define the *bigraded Hilbert series* of W as

$$\mathrm{Hilb}(W; q, t) = \sum_{i,j\geq 0} t^i q^j \dim(W^{(i,j)}), \tag{10.3.27}$$

where the subspaces $W^{(i,j)}$ consist of those elements of W that are bi-homogeneous of degree i in the x variables and j in the y variables, so $W = \bigoplus_{i,j\geq 0} W^{(i,j)}$. Also define the *diagonal action* of S_n on W by

$$\sigma f = f(x_{\sigma_1}, \ldots x_{\sigma_n}, y_{\sigma_1}, \ldots, y_{\sigma_n}), \quad \sigma \in S_n,\ f \in W. \tag{10.3.28}$$

Clearly the diagonal action fixes the subspaces $W^{(i,j)}$, so we can define the bigraded Frobenius series of W as

$$\mathrm{Frob}(W; q, t) = \sum_{i,j\geq 0} t^i q^j \sum_{\lambda \vdash n} s_\lambda \,\mathrm{Mult}\,(\chi^\lambda, W^{(i,j)}). \tag{10.3.29}$$

Similarly, let W^ϵ be the subspace of alternating elements in W, and

$$\mathrm{Hilb}(W^\epsilon; q, t) = \sum_{i,j\geq 0} t^i q^j \dim(W^{\epsilon(i,j)}). \tag{10.3.30}$$

As in the case of subspaces of $\mathbb{C}[X_n]$,

$$\mathrm{Hilb}(W^\epsilon; q, t) = \langle \mathrm{Frob}(W^\epsilon; q, t), s_{1^n}\rangle. \tag{10.3.31}$$

For $\mu \in \mathrm{Par}(n)$, let $(c_1, r_1), \ldots, (c_n, r_n)$ be the $(a'+1, l'+1) = (\text{column}, \text{row})$ coordinates of the cells of μ, taken in some arbitrary order, and let

$$\Delta_\mu(X_n, Y_n) = \left| y_i^{c_j-1} x_i^{r_j-1} \right|_{i,j=1,n}. \qquad (10.3.32)$$

For example,

$$\Delta_{(2,2,1)}(X_5, Y_5) = \begin{vmatrix} 1 & y_1 & x_1 & x_1y_1 & x_1^2 \\ 1 & y_2 & x_2 & x_2y_2 & x_2^2 \\ 1 & y_3 & x_3 & x_3y_3 & x_3^2 \\ 1 & y_4 & x_4 & x_4y_4 & x_4^2 \\ 1 & y_5 & x_5 & x_5y_5 & x_5^2 \end{vmatrix}. \qquad (10.3.33)$$

Note that, up to sign, $\Delta_{1^n}(X_n, 0) = \Delta(X_n)$, the Vandermonde determinant.

For $\mu \vdash n$, let $V(\mu)$ denote the linear span of $\Delta_\mu(X_n, Y_n)$ and its partial derivatives of all orders. Note that, although the sign of Δ_μ may depend on the arbitrary ordering of the cells of μ we started with, $V(\mu)$ is independent of this ordering. Garsia and Haiman conjectured [37] the following result, which was proved by Haiman in 2001 [69].

Theorem 10.3.6 *For all $\mu \vdash n$,*

$$\mathrm{Frob}(V(\mu); q, t) = \tilde{H}_\mu, \qquad (10.3.34)$$

where $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ is the "modified Macdonald polynomial" defined as

$$\tilde{H}_\mu = \sum_{\lambda \vdash n} \tilde{K}_{\lambda,\mu}(q, t) s_\lambda. \qquad (10.3.35)$$

Note that Theorem 10.3.6 implies $\tilde{K}_{\lambda,\mu}(q, t) \in \mathbb{N}[q, t]$.

Corollary 10.3.7 *For all $\mu \vdash n$*, $\dim V(\mu) = n!$.

Remark 10.3.8 Corollary 10.3.7 was known as the "$n!$ *conjecture*". Although Theorem 10.3.6 appears to be much stronger, Haiman proved [68] in the late 1990's that Corollary 10.3.7 implies Theorem 10.3.6.

It is clear from the definition of $V(\mu)$ that

$$\mathrm{Frob}(V(\mu); q, t) = \mathrm{Frob}(V(\mu'); t, q). \qquad (10.3.36)$$

Theorem 10.3.6 thus gives a geometric interpretation to (10.3.25).

Example 10.3.9 It is known that

$$\tilde{H}_{1^n}(X; q, t) = \mathrm{Frob}(V(1^n); q, t) = (t; t)_n h_n \left[\frac{X}{1-t} \right],$$
$$\tilde{H}_n(X; q, t) = (q; q)_n h_n \left[\frac{X}{1-q} \right].$$

Before Haiman proved the general case using algebraic geometry, Garsia and Haiman proved the special case of the $n!$ conjecture when μ is a hook by combinatorial methods [39]. The case where μ is an augmented hook was proved by Reiner [134].

From (10.3.14) we see that

$$\tilde{H}_\mu[X;q,t] = t^{n(\mu)} J_\mu\left[\frac{X}{1-1/t}; q, 1/t\right] \tag{10.3.37}$$
$$= t^{-n} P_\mu\left[\frac{X}{1-1/t}; q, 1/t\right] \prod_{x\in\mu}(t^{l+1} - q^a).$$

Macdonald derived formulas for the coefficients in the expansion of $e_k P_\mu(X;q,t)$ and also $h_k\left[X\frac{1-t}{1-q}\right] P_\mu(X;q,t)$ in terms of the $P_\lambda(X;q,t)$. These expansions reduce to the classical Pieri formulas for Schur functions discussed in Theorem 10.2.5 when $t = q$. When expressed in terms of the J_μ, the h_k Pieri rule becomes [120, Ex. 4, p.363]

$$h_k\left[X\frac{(1-t)}{1-q}\right] J_\mu = \sum_{\substack{\lambda\in\mathrm{Par} \\ \lambda/\mu \text{ is a horizontal } k\text{-strip}}} \frac{\prod_{x\in\mu}(1 - q^{a_\mu + \chi(x\in B)} t^{l_\mu + \chi(x\notin B)})}{\prod_{x\in\lambda}(1 - q^{a_\lambda + \chi(x\in B)} t^{l_\lambda + \chi(x\notin B)})} J_\lambda, \tag{10.3.38}$$

where B is the set of columns which contain a cell of λ/μ, a_μ, l_μ are the values of a, l when the cell is viewed as part of μ, and a_λ, l_λ are the values of a, l when the cell is viewed as part of λ.

The Space of Diagonal Harmonics

Let $p_{h,k}[X_n, Y_n] = \sum_{i=1}^n x_i^h y_i^k$, $h, k \in \mathbb{N}$ denote the “polarized power sum”. It is known that the set $\{p_{h,k}[X_n, Y_n], h + k \geq 0\}$ generate $\mathbb{C}[X_n, Y_n]^{S_n}$, the ring of invariants under the diagonal action. Thus a natural analog of the quotient ring R_n of coinvariants is the quotient ring DR_n of diagonal coinvariants defined by

$$\mathrm{DR}_n = \mathbb{C}[X_n, Y_n]/\left\langle \sum_{i=1}^n x_i^h y_i^k \right\rangle_{h,k\in\mathbb{Z};\, h+k>0}. \tag{10.3.39}$$

By analogy we also define the space of diagonal harmonics DH_n by

$$\mathrm{DH}_n = \left\{ f \in \mathbb{C}[X_n, Y_n] : \sum_{i=1}^n \frac{\partial^h}{\partial x_i^h} \frac{\partial^k}{\partial y_i^k} f = 0,\ h, k \in \mathbb{Z},\ h + k > 0 \right\}. \tag{10.3.40}$$

Many of the properties of H_n and R_n carry over to two sets of variables. For example DH_n is a finite dimensional vector space which is isomorphic to DR_n as an S_n-module (under the diagonal action). The dimension of these spaces turns out to be $(n+1)^{n-1}$, a result which was first conjectured by Haiman [67] and proved by him in 2001 [70]. His proof uses many of the techniques and results from his proof of the $n!$ conjecture. See [143] for a nice expository account of the $n!$ theorem and the $(n+1)^{n-1}$ theorem.

Example 10.3.10 An explicit basis for DH_2 is given by $\{1, x_2 - x_1, y_2 - y_1\}$. The elements $x_2 - x_1$ and $y_2 - y_1$ form a basis for DH_2^ϵ. Thus

$$\mathrm{Frob}(\mathrm{DH}_2; q, t) = s_2 + (q + t)s_{1^2}. \tag{10.3.41}$$

The number $(n + 1)^{n-1}$ is known to count some interesting combinatorial structures. For example, it counts the number of rooted, labelled trees on $n+1$ vertices with root node labelled 0. It also counts the number of parking functions on n cars. In the next section we discuss a conjecture of Haglund and Loehr which gives a combinatorial description for $\mathrm{Hilb}(\mathrm{DH}_n; q, t)$ in terms of statistics on parking functions [56].

We let $M = (1 - q)(1 - t)$ and, for $\mu \in \mathrm{Par}$,

$$T_\mu = t^{n(\mu)} q^{n(\mu')}, \quad w_\mu = \prod_{x \in \mu} (q^a - t^{l+1})(t^l - q^{a+1}). \tag{10.3.42}$$

Haiman derives the $(n + 1)^{n-1}$ result as a corollary of the following formula for the Frobenius series of DH_n.

Theorem 10.3.11 (Haiman, [70])

$$\mathrm{Frob}(\mathrm{DH}_n; q, t) = \sum_{\mu \vdash n} \frac{T_\mu M \tilde{H}_\mu \Pi_\mu B_\mu}{w_\mu}, \tag{10.3.43}$$

where B_μ, Π_μ *are as in* (10.3.7).

Theorem 10.3.11 was conjectured by Garsia and Haiman in [38]. The conjecture was inspired in part by suggestions of C. Procesi.

From (10.3.21) and the fact that $T_\mu = e_n[B_\mu]$, we have $\langle \tilde{H}_\mu, s_{1^n} \rangle = T_\mu$. Thus if we define

$$C_n(q, t) = \sum_{\mu \vdash n} \frac{T_\mu^2 M \Pi_\mu B_\mu}{w_\mu}, \tag{10.3.44}$$

then by (10.3.43),

$$C_n(q, t) = \langle \mathrm{Frob}(\mathrm{DH}_n; q, t), s_{1^n} \rangle = \mathrm{Hilb}(\mathrm{DH}_n^\epsilon; q, t). \tag{10.3.45}$$

For instance, from Example 10.3.10, we have $C_2(q, t) = q + t$. $C_n(q, t)$ is referred to as the q, t-Catalan sequence, since Garsia and Haiman proved that $C_n(1, 1)$ reduces to $C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. The C_n have quite a history and arise very frequently in combinatorics and elsewhere. See [142, Ex. 6.19], [144] for over 210 different objects counted by the Catalan numbers.

The Nabla Operator

We begin this section with a slight generalization of the Koornwinder-Macdonald reciprocity formula, in a form which occurs in [43].

Theorem 10.3.12 *Let* $\mu, \lambda \in \mathrm{Par}$, $z \in \mathbb{R}$. *Then*

$$\frac{\tilde{H}_\mu[1 + z(MB_\lambda - 1); q, t]}{\prod_{x \in \mu}(1 - zq^{a'} t^{l'})} = \frac{\tilde{H}_\lambda[1 + z(MB_\mu - 1); q, t]}{\prod_{x \in \lambda}(1 - zq^{a'} t^{l'})}. \tag{10.3.46}$$

Proof (Sketch) By cross multiplying, we can rewrite (10.3.46) as a statement saying two polynomials in z are equal. Letting $z = t^n$ for $n \in \mathbb{N}$, the two polynomials agree by (10.3.9) and two polynomials which agree on infinitely many values must be equal. □

Remark 10.3.13 If $|\mu|, |\lambda| > 0$, then we can cancel the factor of $1 - z$ in the denominators on both sides of (10.3.46) and then set $z = 1$ to obtain

$$\frac{\tilde{H}_\mu[MB_\lambda; q, t]}{\Pi_\mu} = \frac{\tilde{H}_\lambda[MB_\mu; q, t]}{\Pi_\lambda}. \tag{10.3.47}$$

Another useful special case of (10.3.46) is $\lambda = \emptyset$, which gives

$$\tilde{H}_\mu[1 - z; q, t] = \prod_{x \in \mu} (1 - zq^{a'} t^{l'}). \tag{10.3.48}$$

Let ∇ be the linear operator on symmetric functions which satisfies

$$\nabla \tilde{H}_\mu = T_\mu \tilde{H}_\mu. \tag{10.3.49}$$

It turns out that many of the results in Macdonald polynomials and diagonal harmonics can be elegantly expressed in terms of ∇. Some of the basic properties of ∇ were first worked out by F. Bergeron [10], and more advanced applications followed in a series of papers by Bergeron, Garsia, Haiman and Tesler [15], [14], [43].

Proposition 10.3.14

$$\nabla e_n = \sum_{\mu \vdash n} \frac{T_\mu M \tilde{H}_\mu \Pi_\mu B_\mu}{w_\mu}, \qquad n > 0. \tag{10.3.50}$$

Hence Theorem 10.3.11 *is equivalent to*

$$\mathrm{Frob}(\mathrm{DH}_n; q, t) = \nabla e_n. \tag{10.3.51}$$

Proof (Sketch) Expressing (10.3.11) in terms of the $\tilde{H}_\mu$ and using some simple plethystic substitutions, (10.3.14) is equivalent to

$$e_n\left[\frac{XY}{M}\right] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t] \tilde{H}_\mu[Y; q, t]}{w_\mu}. \tag{10.3.52}$$

Now let $Y = M$, use (10.3.47), and then apply ∇ to both sides. □

10.4 The Combinatorics of the Space of Diagonal Harmonics

The Parking Function Model

A *Dyck path* is a lattice path in the first quadrant of the xy-plane from $(0, 0)$ to (n, n) consisting of unit north N and east E steps which never goes below the diagonal $x = y$. We let $L^+_{n,n}$ denote the set of all such Dyck paths. A *parking function* σ is a placement of the integers $1, 2, \ldots, n$ (called "cars") just to the right of the N steps of a Dyck path, in such a way that the numbers are strictly decreasing down columns. The *reading word* of σ is the permutation obtained by reading the cars along diagonals in a SW direction, outside to in, as in Figure 10.5. To a given parking function σ, we associate two statistics area(σ) and dinv(σ). The area

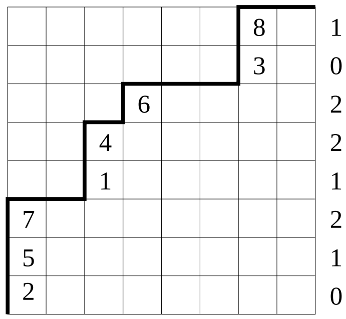

Figure 10.5 A parking function with area = 9, dinv = 6, and reading word 64781532.

statistic is defined as the number of squares strictly below the Dyck path and strictly above the diagonal. The dinv statistic is the number of pairs of cars which form either "primary" or "secondary" inversions. Pairs of cars form a primary inversion if they are in the same diagonal, with the larger car in a higher row. Pairs form a secondary inversion if they are in successive diagonals, with the larger car in the outer diagonal and in a lower row. For example, for the parking function in Figure 10.5, car 8 forms primary inversions with cars 1 and 5, while car 5 forms a secondary inversion with car 3. The set of inversion pairs for this parking function is $\{(6,4),(7,1),(8,1),(8,5),(5,3),(3,2)\}$, so dinv $= 6$ while area $= 9$. A conjecture of Loehr and the author expresses $\mathrm{Hilb}(\mathrm{DH}_n;q,t)$ as a positive sum of monomials, one for each parking function.

Conjecture 10.4.1 ([56], [53, Chap. 5])

$$\mathrm{Hilb}(\mathrm{DH}_n;q,t) = \sum_{\sigma} q^{\mathrm{dinv}(\sigma)} t^{\mathrm{area}(\sigma)}, \tag{10.4.1}$$

where the sum is over all parking functions with n cars.

Remark 10.4.2 Armstrong [5] has recently introduced a hyperplane arrangement model for $\mathrm{Hilb}(\mathrm{DH}_n;q,t)$ involving a pair of hyperplane arrangements with a statistic associated to each one. See also [6]. He gives a bijection with parking functions which sends his pair of hyperplane arrangement statistics to (area′, bounce), another pair of statistics which Haglund and Loehr showed have the same distribution over parking functions as (dinv, area).

Haglund, Haiman, Loehr, Remmel, and Ulyanov [58] introduced a generalization of Conjecture 10.4.1 which gives a combinatorial formula for the coefficient of a monomial symmetric function in the character ∇e_n. To describe it, let β, γ be two compositions with $|\beta| + |\gamma| = n$. Then their conjecture can be expressed as:

Conjecture 10.4.3 [58], [53, Chap. 6] (see Section 10.10 for a discussion of recent work on this conjecture)

$$\langle \nabla e_n, e_\beta h_\gamma \rangle = \sum_{\sigma:\ \text{read}(\sigma) \text{ is an } \gamma,\beta\text{-shuffle}} q^{\text{dinv}(\sigma)} t^{\text{area}(\sigma)}, \tag{10.4.2}$$

where the sum is over all parking functions σ on n cars whose reading word is a shuffle of increasing sequences $(1, 2, \cdots, \gamma_1), (\gamma_1 + 1, \gamma_1 + 2, \cdots, \gamma_1 + \gamma_2), \ldots,$ and decreasing sequences $(n, n-1, \cdots, n-\beta_1+1), (n-\beta_1, n-\beta_1-1, \cdots, n-\beta_1-\beta_2+1), \ldots$. (Here a shuffle of two sequences A and B is a permutation where all the elements of A occur in order, and all the elements of B occur in order, but the elements of A and B are intertwined in an arbitrary manner.)

This conjecture is commonly known as the *Shuffle Conjecture* due to its statement involving shuffles of sequences. If $\beta = (n)$, the Shuffle Conjecture reduces to a theorem of Garsia and Haglund [35], [36], [53, Chap. 3], which gives a combinatorial formula for the sign character of DH_n, or equivalently for the rational function $C_n(q,t)$ defined in (10.3.44). More generally, if $\beta = (n-d), \gamma = (d)$, the Shuffle Conjecture reduces to the (q,t)-Schröder Theorem of Haglund [51], which can be described in terms of sums over "Schröder lattice paths" consisting of north, east, and diagonal steps. If $\gamma = 1^n$ then the Shuffle Conjecture reduces to Conjecture 10.4.1.

The Shuffle Conjecture can also be expressed in the following way.

Conjecture 10.4.4 (Alternate Form of the Shuffle Conjecture)

$$\nabla e_n = \sum_{\pi \in L^+_{n,n}} t^{\text{area}(\pi)} \mathcal{F}_\pi(X;q), \tag{10.4.3}$$

where the sum is over all Dyck paths π and

$$\mathcal{F}_\pi(X;q) = \sum_{\sigma \in \text{PF}(\pi)} q^{\text{dinv}(\sigma)} F_{n,\text{Des}(\text{read}(\sigma)^{-1})}(X). \tag{10.4.4}$$

Here $\text{PF}(\pi)$ denotes the set of all parking functions for the Dyck path π. For example, for the parking function of Figure 10.5, the inverse descent set of the reading word is $\{2,3,5\}$, so in (10.4.4) this parking function would be weighted by $q^6 F_{8,\{2,3,5\}}$.

It is not at all obvious that the right-hand-side of (10.4.3) is a symmetric function, but in fact each of the $\mathcal{F}_\pi(X;q)$ are symmetric functions. The proof of this relies on the theory of LLT polynomials, which were introduced by Lascoux, Leclerc, and Thibon in 1997 [103]. Their original construction is described in terms of ribbon tableaux, but we will present an equivalent formulation due to M. Haiman and his student Michelle Bylund, which is presented in the appendix of [57]. We start with a grid of dotted lines, all of slope 1, with the vertical distance between successive dotted lines equaling 1. We then embed a tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ of skew shapes in this grid, so that each square has one of the dotted lines as a diagonal of the square, and we fill each γ_i with a SSYT T_i of that shape. In the example of Figure 10.6, the tuple is $(22, 22, 11)$. If $\mathbf{T} = (T_1, T_2, \ldots, T_k)$ denotes such a tuple of SSYT, we let $\text{inv}(\mathbf{T})$

denote the number of "inversion pairs" of T. An inversion pair is a pair of integers a, b, with $b > a$, a, b in different skew shapes γ_i, and either

- a, b are on the same diagonal, with b in a column strictly left of a, or
- a, b are on successive diagonals, with b strictly NE of a, i.e. b is in the diagonal just above that containing a, and in a column strictly to the right of the one containing a.

For example, for the tuple $\mathbf{T}$ in Figure 10.6, the 5 above the 3 in γ_1 forms inversion pairs of the first type with the 1 and 3 from γ_2, and also with the 2 in γ_3, while the 6 forms inversion pairs with that same 5, 1, and 3, and also the 2 from γ_1.

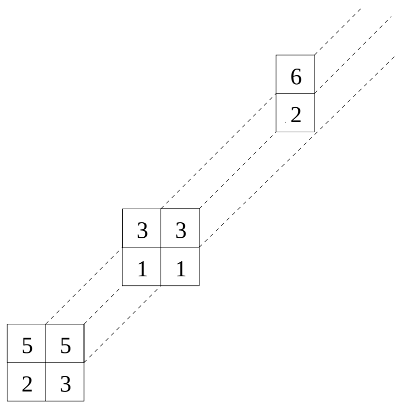

Figure 10.6 A tuple of SSYT occurring in the definition of an LLT product of the shapes $(22, 22, 11)$. This tuple has 13 inversion pairs.

Theorem 10.4.5 (Lascoux-Leclerc-Thibon [103], Haiman-Bylund [57])
Given any tuple of skew shapes $(\gamma_1, \gamma_2, \ldots, \gamma_k)$, *the sum*

$$\sum_{\mathbf{T}=(T_1,\ldots,T_k),\ T_i \in SSYT(\gamma_i)} q^{\mathrm{inv}(\mathbf{T})} x^{\mathbf{T}} \tag{10.4.5}$$

is a symmetric function. Here $x^{\mathbf{T}}$ *is the product* $x^{T_1} x^{T_2} \cdots x^{T_k}$ *of the usual* x*-weights of the SSYT.*

We will refer to the symmetric function (10.4.5) as the LLT product of the γ_i. If the multiset of numbers contained in the tuple $\mathbf{T}$ is just the set $\{1, 2, \ldots, n\}$, we say $\mathbf{T} \in \mathrm{SYT}(\gamma)$. Let the reading word read($\mathbf{T}$) of an LLT tuple of SSYT be the word obtained by reading along diagonals, outside to in, and in a NE direction along each diagonal, so for example the reading word of the tuple in Figure 10.6 is 5362513231. Furthermore let $\zeta(\mathbf{T})$ denote the (unique) element of $\mathrm{SYT}(\gamma)$ whose reading word is the same as the standardization of the reading word of the tuple $\mathbf{T}$. Then clearly $\mathrm{inv}(\mathbf{T}) = \mathrm{inv}(\zeta(\mathbf{T}))$, so by Remark 10.2.8 we have:

Corollary 10.4.6 *The LLT product of the γ_i in* (10.4.5) *can be expressed as*

$$\sum_{\mathbf{T}=(T_1,\ldots,T_k)\in SYT(\gamma)} q^{\mathrm{inv}(\mathbf{T})} F_{n,\mathrm{Des}(\mathrm{read}(\mathbf{T}))^{-1})}(X). \tag{10.4.6}$$

Corollary 10.4.7 *The function $\mathcal{F}_\pi(X;q)$ from* (10.4.4) *is a symmetric function, since it is the LLT product of vertical strips.*

Proof For each parking function σ occurring in the definition of F_π, there is a corresponding element $\mathbf{T}(\sigma) \in \mathrm{SYT}(\gamma)$ where $\mathrm{read}(\sigma) = \mathrm{read}(\mathbf{T}(\sigma))$, with the same set of inversion pairs. (See Figure 10.7 for an example.) Now use Corollary 10.4.6. □

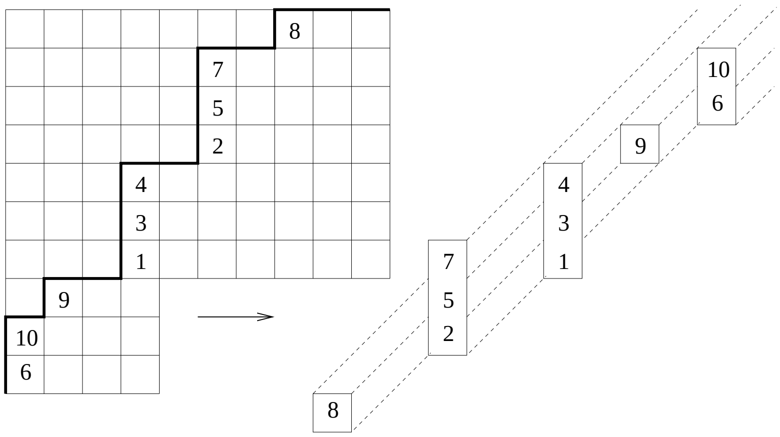

Figure 10.7 A SYT in the LLT product of vertical strips $(1^3/1^2, 1^4/1, 1^3, 1^2/1, 1^2)$ with 5 inversion pairs.

Lascoux, Leclerc, and Thibon conjectured that any LLT product is Schur positive, i.e. when expressed in the Schur basis the coefficients are in $\mathbb{N}[q]$. Note that if $q = 1$, the LLT product of the γ_i is just the product of Schur functions $s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_k}$, which is Schur-positive by the Littlewood-Richardson rule. By results of Leclerc and Thibon [104], and Kashiwara and Tanisaki [82], this conjecture is known to be true for the case where each γ_i is a partition (i.e. non-skew) shape, with the lower-left-hand square of each γ_i all on the same diagonal. In [58] this result is extended somewhat to include the $\mathcal{F}_\pi(X;q)$. In 2007 Grojnowski and Haiman announced a proof of the general case, which uses Kazhdan-Lusztig theory [48].

Remark 10.4.8 Haglund, Morse, and Zabrocki [63] have introduced a refinement of the Shuffle Conjecture known as the Compositional Shuffle Conjecture, which says the portion of the right-hand-side of (10.4.3) involving paths π which hit the main diagonal $x = y$ at touch points $(a_1, a_1), (a_1 + a_2, a_1 + a_2), \ldots, (n, n)$, can be expressed as ∇ applied to a "compositional Hall-Littlewood" polynomial. These generalized Hall-Littlewood polynomials are defined using Jing operators; if the composition $(a_1, a_2, \ldots)$ is a partition μ they reduce to $(-1/q)^n \tilde{H}_{\mu'}(X; q, 0)$. Garsia, Xin, and Zabrocki [33], [44] have used manipulations of plethystic Macdonald polynomial identities and bijective results of Garsia's student Angela Hicks

[73] to prove many special cases of this conjecture; in particular they obtain a "Compositional (q,t)-Schröder Theorem".

Remark 10.4.9 In his original work on diagonal harmonics, Haiman introduced a more general space $\mathrm{DH}_n^{(k)}$, where k is a positive integer. When $k = 1$ it reduces to DH_n. Haiman [70] proved that $\mathrm{Frob}(\mathrm{DH}_n^{(k)}) = \nabla^k e_n$, and most of the combinatorial conjectures in this section have "parameter k" versions, involving lattice paths in an $n \times kn$-rectangle which never go below the diagonal $x = ky$. In particular, there is a parameter k version of the Shuffle Conjecture [58], but at this writing even the sign-character case of this remains open.

Recently a dramatic generalization of the Shuffle Conjecture, and also the extension of it discussed in Remark 10.4.9, has been introduced. Many different researchers played a role in its formulation. It depends on a pair (m,n) of relatively prime positive integers. The combinatorial side of this conjecture occurs both in work of Hikita [74], and unpublished work of D. Armstrong, and is fully described in [46] and also [34].

Let $\mathrm{Grid}(m,n)$ be the $n \times m$ grid of labelled squares whose upper-left-hand corner square is labelled with $(n-1)(m-1)-1$, and whose labels decrease by m as you go down columns and by n as you go across rows. For example,

$$\mathrm{Grid}(3,7) = \begin{array}{|l|c|r|}\hline 11 & 4 & -3 \\ \hline 8 & 1 & -6 \\ \hline 5 & -2 & -9 \\ \hline 2 & -5 & -12 \\ \hline -1 & -8 & -15 \\ \hline -4 & -11 & -18 \\ \hline -7 & -14 & -21 \\ \hline \end{array} \tag{10.4.7}$$

To the corners of the squares of $\mathrm{Grid}(m,n)$ we associate Cartesian coordinates, where the lower-left-hand corner of the grid has coordinates $(0,0)$, and the upper-right-hand-corner of the grid (m,n). An (m,n)-Dyck path is a lattice path of unit N and E steps from $(0,0)$ to (m,n) which never goes below the line $nx = my$, and we denote the set of such paths by $L^+_{(m,n)}$. One finds that $L^+_{(m,n)}$ is the same as the set of lattice paths π from $(0,0)$ to (m,n) for which none of the squares with negative labels are above π. For a given π, we let $\mathrm{area}(\pi)$ denote the number of squares in $\mathrm{Grid}(m,n)$ with positive labels which are below π. Furthermore, let $\mathrm{dinv}(\pi)$ denote the number of squares in $\mathrm{Grid}(m,n)$ which are above π and whose arm and leg lengths satisfy

$$\frac{a}{l+1} < \frac{m}{n} < \frac{a+1}{l}. \tag{10.4.8}$$

Here by the arm of a square s we mean the number of squares in the same row as s, to the right of s, and to the left of π. The leg of s is the number of squares below s and in its column, and above π. For example, if $(m,n) = (3,7)$ and $\pi = NNNNNEENNE$, then $\mathrm{area}(\pi) = 2$ (corresponding to the squares with labels 2 and 5), and $\mathrm{dinv}(\pi) = 2$; the squares with labels $11, 8, 4, 1$ have $a = l = 1; a = 1, l = 0; a = 0, l = 1; a = l = 0$, respectively, and so the squares with labels 8 and 11 do not satisfy (10.4.8), while the squares with labels 1 and 4 do.

Let an (m, n)-parking function be a path $\pi \in L^+_{(m,n)}$ together with a placement of the integers 1 through n (called *cars*) just to the right of the N steps of π, with strict decrease down columns. For such a pair P, for $1 \le j \le n$ we let rank(j) be the label of the square that contains car j, and we set

$$\operatorname{tdinv}(P) = \left|\{(i, j) : 1 \le i < j \le n \text{ and } \operatorname{rank}(i) < \operatorname{rank}(j) < \operatorname{rank}(i) + m\}\right|. \tag{10.4.9}$$

Furthermore define the reading word read(P) be the permutation obtained by listing the cars by decreasing order of their ranks. For example, for the $(3, 7)$-parking function of Figure 10.8, tdinv = 3, with inversion pairs formed by pairs of cars $(6, 7)$, $(4, 6)$, and $(2, 4)$, and the reading word is 7642531.

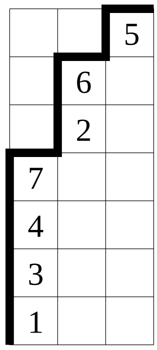

Figure 10.8 A $(3, 7)$-parking function.

Let maxtdinv(π) be tdinv of the parking function for π whose reading word is the reverse of the identity, and for any parking function P for π set

$$\operatorname{dinv}(P) = \operatorname{dinv}(\pi) + \operatorname{tdinv}(P) - \operatorname{maxtdinv}(\pi). \tag{10.4.10}$$

We remark that given a $\pi \in L_{n+1,n}$ and $\sigma \in \mathrm{PF}(\pi)$, the definitions of the dinv and area statistics given above are the same as the ones given by our original definition, if we simply remove the last E step to form a path $\pi' \in L^+_{n,n}$, and view σ as a parking function for π'. Furthermore, the reading word of σ is also the same in both contexts. Hence, the above construction reduces to the original when $m = n + 1$.

There is an amazing extension of the symmetric function ∇e_n which has emerged from work of Burban, Schiffman, Vasserot, Negut and others [20], [138], [139], [130] on the Elliptic Hall algebra and other related objects in algebraic geometry and string theory. Bergeron, Garsia, Leven, and Xin [16] have given a concrete description of the construction of these symmetric functions, by means of a family of plethystic operators on symmetric functions $Q_{(m,n)}$, defined recursively below. They satisfy $Q_{(m+n,n)}(-1)^n = \nabla Q_{(m,n)}(-1)^n$, where the $(-1)^n$ in these relations indicates they are applied to the constant $(-1)^n$, viewed as an element of Λ. Furthermore $Q_{(kn+1,n)}(-1)^n = \nabla^k e_n$, so they contain ∇e_n and the symmetric functions from Remark 10.4.9 as special cases.

To construct the $Q_{(m,n)}$, first let D_k be the operator on symmetric functions $F(X)$ defined by

$$D_k F[X] = F\left[X + \frac{M}{z}\right] \sum_{i \geq 0} (-z)^i e_i[X] \quad \Big|_{z^k}, \tag{10.4.11}$$

where $|_{z^k}$ means "take the coefficient of z^k in ", and again $M = (1-q)(1-t)$. The D_k operators were introduced in [43]; they form important building blocks in the development of plethystic identities involving Macdonald polynomials. We require the following Proposition.

Proposition 10.4.10 [17]
For any coprime pair of integers (m,n) with $m,n > 1$, there is a unique pair (a,b) satisfying

$$(1) \quad 1 \leq a \leq m-1, \qquad (2) \quad 1 \leq b \leq n-1, \qquad (3) \quad mb + 1 = na. \tag{10.4.12}$$

Let $c = m - a$ and $d = n - b$. Then both (a,b) and (c,d) are coprime pairs.

For coprime $m,n > 1$ write $\mathrm{Split}(m,n) = (a,b) + (c,d)$, and otherwise set

$$(a) \quad \mathrm{Split}(1,n) = (1,n-1) + (0,1), \qquad (b) \quad \mathrm{Split}(m,1) = (1,0) + (m-1,1). \tag{10.4.13}$$

If $\mathrm{Split}(m,n) = (a,b) + (c,d)$, recursively set

$$Q_{(m,n)} = \frac{1}{M} \left[Q_{(c,d)}, Q_{(a,b)} \right] \tag{10.4.14}$$

with base cases $Q_{(1,0)} = D_0$ and $Q_{(0,1)} = -\underline{e}_1$. Here $\underline{e}_1$ is multiplication by e_1 and $[x,y] = xy - yx$.

Conjecture 10.4.11 (The Rational Shuffle Conjecture [46], [34])
For any pair of relatively prime positive integers (m,n) and any pair of compositions α, β with $\sum_i \alpha + \sum_j \beta = n$, we have

$$\langle Q_{(m,n)}(-1)^n, e_\alpha h_\beta \rangle = \sum_{\substack{(m,n) \text{ parking functions } P \\ \mathrm{read}(P) \text{ is an } \alpha,\beta \text{ shuffle}}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\pi)}, \tag{10.4.15}$$

where the sum is over all (m,n) parking functions P whose reading word is a shuffle of decreasing sequences of lengths $\alpha_1, \alpha_2, \ldots$ and increasing sequences of lengths $\beta_1, \beta_2, \ldots$.

An alternate formulation of Conjecture 10.4.11 is

$$Q_{(m,n)}(-1)^n = \sum_{\pi \in L^+_{m,n}} t^{\mathrm{area}(\pi)} \mathcal{F}_\pi(X;q), \tag{10.4.16}$$

where

$$\mathcal{F}_\pi(X;q) = \sum_{\sigma \in \mathrm{PF}(\pi)} q^{\mathrm{dinv}(\sigma)} F_{n,\mathrm{Des}(\mathrm{read}(\sigma)^{-1})}(X). \tag{10.4.17}$$

We leave it as an exercise for the interested reader to show that for any $\pi \in L^+_{(m,n)}$, $\mathcal{F}_\pi(X;q)$ is an LLT product of vertical strips, times a power of q. Hence the right-hand-side of (10.4.16) is a Schur positive symmetric function.

Remark 10.4.12 The right-hand-side of (10.4.16) first arises in work of Hikita [74], who proved it is the bigraded Frobenius series of a certain module arising from affine Springer fibers. When $m = n + 1$ though this module is not obviously isomorphic to DH_n.

Example 10.4.13 When $t = 1/q$, we have [34]

$$q^{(m-1)(n-1)/2} Q_{(m,n)}(-1)^n = \frac{1}{[m]_q} e_n[X[m]_q], \tag{10.4.18}$$

where $[m]_q = (1 - q^m)/(1 - q)$. As a special case we have, when $t = 1/q$,

$$q^{(m-1)(n-1)/2} \langle Q_{(m,n)}(-1)^n, s_{1^n} \rangle = \begin{bmatrix} n+m-1 \\ n \end{bmatrix}, \tag{10.4.19}$$

where $\begin{bmatrix} n+m-1 \\ n \end{bmatrix}$ is the q-binomial coefficient. D. Stanton has asked for a statistic qstat which would allow us to express the right-hand-side of (10.4.19) as a sum of q^{stat} over (m, n)-Dyck paths. In the case $m = n + 1$ MacMahon [122, p. 214] proved that you can generate this using qstat = maj, where to compute maj on a Dyck path π you write π as a sequence of N and E steps, replace each E by a 1 and each N by a 0, then take the usual maj statistic on words. For general (m, n) there is no known variant of maj which works, but if we assume Conjecture 10.4.15 we can use qstat = dinv $+(m - 1)(n - 1)/2 -$ area.

The Superpolynomial Knot Invariant

In 2006 Dunfield, Gukov, and Rasmussen [27] hypothesized the existence of a three-parameter knot invariant, now known as the "superpolynomial knot invariant" of a knot K, denoted $\mathcal{P}_K(a, q, t)$, which includes the HOMFLY polynomial as a special case. Since then various authors proposed different possible definitions of the superpolynomial, which are conjecturally all equivalent. These definitions typically involve homology though, and are difficult to compute.

Let (m, n) be a pair of relatively prime positive integers, and let $T_{(m,n)}$ denote the (m, n) torus knot, which is the knot obtained by wrapping a string around the torus at an angle such that by the time you return to the starting point, you have wrapped around m times in one direction and n in the other. An accepted definition of $\mathcal{P}_{T_{(m,n)}}(a, q, t)$ has emerged from work of Aganagic and Shakirov [1], [2] (using refined Chern-Simons theory) and Cherednik [25] (using the double affine Hecke algebra). Gorsky and Negut [46] showed that these two different constructions yield the same three parameter knot invariant which is now accepted as the definition of the superpolynomial for torus knots. Let

$$\tilde{\mathcal{P}}_{(m,n)}(a, q, t) = q^{(m-1)(n-1)} \mathcal{P}_{T_{(m,n)}}(a, 1/q, 1/t) \tag{10.4.20}$$

denote the "modified superpolynomial" of $T_{(m,n)}$. It can be described analytically as

$$\sum_{d=0}^{n} (-a)^d \langle Q_{(m,n)}(-1)^n, e_{n-d} h_d \rangle, \tag{10.4.21}$$

which is basically the generating function for hook shapes for the symmetric function occurring in Conjecture 10.4.15. Thus if we assume this conjecture we have a nice positive expression for the modified superpolynomial for torus knots, which E. Gorsky has pointed out is equivalent to a conjectured formula for it from [131]. See also [55], [47].

Remark 10.4.14 In [34] the "Compositional Rational Shuffle Conjecture" is introduced, which contains both the Rational Shuffle Conjecture and the Compositional Shuffle Conjecture as special cases. The key element in the conjecture is a subtle construction of $Q_{(m,n)}$ operators for non-relatively prime (m, n).

Tesler Matrices and a Polynomial Formula for the Hilbert Series of DH_n

Haglund [54] obtained a new polynomial expression for $\mathrm{Hilb}(\mathrm{DH}_n)$. The expression is in $\mathbb{Z}[q, t]$, and has some negative coefficients, but hopefully further work will lead to a positive expression as in (10.4.1). A *Tesler matrix* of order n is an $n \times n$ upper-triangular matrix of nonnegative integers, such that for any j in the range $1 \le j \le n$, the sum of all the entries in the jth row of the matrix, minus the sum of all the entries in the jth column strictly above the diagonal, equals 1. Letting Q_n denote the set of Tesler matrices of order n, the elements of Q_3 are

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&0&2\end{bmatrix}, \begin{bmatrix}0&1&0\\0&2&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\0&1&1\\0&0&2\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&2\\0&0&3\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\0&0&2\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&1\\0&0&3\end{bmatrix}.$$

Let $[k]_{q,t} = (t^k - q^k)/(t - q)$ denote the (q, t)-analog of the integer k, and recall that $M = (1 - q)(1 - t)$. To each Tesler matrix C we associate the weight

$$\mathrm{wt}(C) = (-M)^{\mathrm{pos}(C)-n} \prod_{c_{ij}>0} [c_{ij}]_{q,t},$$

where $\mathrm{pos}(C)$ is the number of positive entries in C. For example, the weight of

$$\begin{bmatrix}0&1&0\\0&1&1\\0&0&2\end{bmatrix}$$

is $(t + q)(-M) = -(t + q)(1 - q)(1 - t)$.

Theorem 10.4.15 [54]

$$\mathrm{Hilb}(\mathrm{DH}_n) = \sum_{C \in Q_n} \mathrm{wt}(C). \tag{10.4.22}$$

Example 10.4.16 When $n = 3$ (10.4.22) becomes

$$\begin{aligned}\mathrm{Hilb}(\mathrm{DH}_3) = 1 &+ (t + q) + (t + q) - (1 - q)(1 - t)(t + q)\\ &+ (t + q)(t^2 + tq + q^2) + (t + q) + (t^2 + tq + q^2).\end{aligned}$$

Note. Formula (10.4.22) is clearly a polynomial. One advantage it has over (10.4.1) is that it is also clearly symmetric in q, t, in fact is a sum of Schur functions in the set of variables $\{q, t\}$. It is known [13] that $\mathrm{Hilb}(DH_n; q, t)$ is a sum of terms of this form, and more generally so is $\langle \mathrm{Frob}(DH_n; q, t), s_\lambda \rangle$ for any λ, but there is no known combinatorial description of these coefficients. Since $-M = -1 + s_1(q, t) - s_{1,1}(q, t)$, and $[k]_{q,t} = s_{k-1}(q, t)$, (10.4.22) together with the Littlewood-Richardson rule can be used to obtain an expression for $\mathrm{Hilb}(DH_n; q, t)$ as an alternating sum of $s_\lambda(q, t)$. One approach to to the problem would be to study how negative terms cancel in this sum.

Remark 10.4.17 Formula (10.4.22) generalizes easily to a formula for $\mathrm{Hilb}(\mathrm{DH}_n^{(k)})$, by summing over matrices whose hook-sums are $1, k, k, \ldots, k$, and using the same weights $\mathrm{wt}(C)$ [54].

More Recent Work Involving Tesler Matrices

In [46] Gorsky and Negut prove the following.

Theorem 10.4.18 (Gorsky, Negut [46])
For any pair of positive, relatively prime integers (m, n),

$$Q_{m,n}(-1)^n = \sum_{C \in Q_n^{(m)}} \prod_{i=1}^{n} e_{c_{ii}} \prod_{1 \le i < n} \Big([c_{i,i+1} + 1]_{q,t} - [c_{i,i+1}]_{q,t}\Big) \prod_{2 \le i+1 < j \le n} (-M)[c_{i,j}]_{q,t}. \quad (10.4.23)$$

As a corollary, they obtain the following formula for the modified superpolynomial of the (m, n) torus knot from (10.4.20):

Corollary 10.4.19 (Gorsky, Negut [46])
For any pair of positive, relatively prime integers (m, n),

$$\tilde{\mathcal{P}}_{(m,n)}(a, q, t) = \sum_{C \in Q_n^{(m)}} \prod_{\substack{1 \le i \le n \\ c_{i,i} > 0}} (1 - a) \prod_{1 \le i < n} \Big([c_{i,i+1} + 1]_{q,t} - [c_{i,i+1}]_{q,t}\Big) \prod_{2 \le i+1 < j \le n} (-M)[c_{i,j}]_{q,t}.$$
$$(10.4.24)$$

Garsia and Haglund derive a Tesler matrix expression for ∇e_n [32] which is different than the $m = n + 1$ case of (10.4.23). In [128], [129] Mészáros, Morales, and Rhoades introduce the "Polytope of Tesler Matrices", whose points with integer coordinates are in bijection with Tesler matrices. Connections between DH_n and polytopes are further developed in [113]. Wilson [148] obtains Tesler matrix formulas for a broad class of functions of the form $\langle F, h_1^n \rangle$.

10.5 The Expansion of the Macdonald Polynomial into Monomials

In this section we give a combinatorial description of $\tilde{H}_\mu(X; q, t)$ and discuss its consequences. Let $\mu \vdash n$. We let $\mathrm{dg}(\mu)$ denote the "augmented" diagram of μ, consisting of μ together with a row of squares below μ, referred to as the *basement*, with coordinates $(j, 0)$, $1 \le j \le \mu_1$. Define a *filling* σ of μ to be an assignment of a positive integer to each square of μ. For $s \in \mu$, we let $\sigma(s)$ denote the integer assigned to s, i.e the integer occupying s. Let the *reading word* $\mathrm{read}(\sigma) = \sigma_1\sigma_2\cdots\sigma_n$ be the word obtained by reading the occupants of μ across rows left to right, starting with the top row and working downwards. Note that the reading word does not include any of the entries in the basement. In this section we assume the basement is occupied by virtual infinity symbols, i.e. $\sigma(j,0) = \infty$.

For each filling σ of μ we associate x, q and t weights. The x weight is defined in a similar fashion to SSYT, namely

$$x^{\sigma} = \prod_{s\in\mu} x_{\sigma(s)}. \tag{10.5.1}$$

For $s \in \mu$, let North(s) denote the square of μ right above s (if it exists) in the same column, and South(s) the square of dg(μ) directly below s, in the same column. Let the descent set of σ, denoted $\mathrm{Des}(\sigma,\mu)$, be the set of squares $s \in \mu$ where $\sigma(s) > \sigma(\mathrm{South}(s))$. (In this section we regard the basement as containing virtual infinity symbols, so no square in the bottom row of σ can be in $\mathrm{Des}(\sigma,\mu)$). Finally set

$$\mathrm{maj}(\sigma,\mu) = \sum_{s\in\mathrm{Des}(\sigma,\mu)} (\mathrm{leg}(s)+1). \tag{10.5.2}$$

Note that $\mathrm{maj}(\sigma, 1^n) = \mathrm{maj}(\mathrm{read}(\sigma))$, where maj is the usual major index statistic, defined as the sum of those i for which $\sigma_i > \sigma_{i+1}$.

We say a square $u \in \mu$ *attacks* all other squares $v \in \mu$ in its row and strictly to its right, and all other squares $v \in dg(\mu)$ in the row below and strictly to its left. We say u, v attack each other if u attacks v or v attacks u. An *inversion pair* of σ is a pair of squares u, v where u attacks v and $\sigma(u) > \sigma(v)$. Let $\mathrm{invset}(\sigma,\mu)$ denote the set of inversion pairs of σ, $\mathrm{inv}(\sigma,\mu) = |\mathrm{invset}(\sigma,\mu)|$ its cardinality and set

$$\mathrm{inv}(\sigma,\mu) = \mathrm{inv}(\sigma,\mu) - \sum_{s\in\mathrm{Des}(\sigma,\mu)} \mathrm{arm}(s). \tag{10.5.3}$$

For example, if σ is the filling on the left in Figure 10.9 then

$$\begin{aligned}
\mathrm{Des}(\sigma) &= \{(1,2),(1,4),(2,3),(3,2)\},\\
\mathrm{maj}(\sigma) &= 3+1+2+1 = 7,\\
\mathrm{invset}(\sigma) &= \Big|\{\{(1,4),(2,4)\},\{(2,4),(1,3)\},\{(2,3),(1,2)\},\{(1,2),(3,2)\},\\
&\qquad \{(2,2),(3,2)\},\{(2,2),(1,1)\},\{(3,2),(1,1)\},\{(2,1),(3,1)\},\{(2,1),(4,1)\}\}\Big|,\\
\mathrm{inv}(\sigma) &= 9-(2+1+0+0) = 6.
\end{aligned}$$

Note that $\mathrm{inv}(\sigma,(n)) = \mathrm{inv}(\mathrm{read}(\sigma))$, where inv is the usual inversion statistic on words.

Definition 10.5.1 For $\mu \vdash n$, let

$$C_{\mu}(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}^+} x^{\sigma} t^{\mathrm{maj}(\sigma,\mu)} q^{\mathrm{inv}(\sigma,\mu)}. \tag{10.5.4}$$

We define the standardization of a filling σ, denoted $\zeta(\sigma)$, to be the standard filling whose

6	4		
2	4		
3	3	2	
1	4	1	1

11	8		
4	9		
6	7	5	
1	10	2	3

Figure 10.9 On the left, a filling of $(4,3,2,2)$ with reading word 64243321411 and on the right, its standardization.

reading word is the standardization of read(σ). Figure 10.9 gives an example of this. It is immediate from Definition 10.5.1 and Remark 10.2.8 that

$$C_\mu(X;q,t) = \sum_{\tau\in S_n} t^{\mathrm{maj}(\tau,\mu)} q^{\mathrm{inv}(\tau,\mu)} F_{n,\mathrm{Des}(\tau^{-1})}(X), \tag{10.5.5}$$

where we identity a permutation τ with the standard filling whose reading word is τ.

Remark 10.5.2 There is another way to view $\mathrm{inv}(\sigma,\mu)$ which will prove useful to us. Call three squares u, v, w, with $u, v \in \mu$, $w = \mathrm{South}(u)$, and with v in the same row as u and strictly to the right of μ, a *triple*. Given a standard filling σ, we define an orientation on such a triple by starting at the square, either u, v or w, with the smallest element of σ in it, and going in a circular motion, towards the next largest element, and ending at the largest element. We say the triple is an *inversion triple* or a *coinversion triple* depending on whether this circular motion is counterclockwise or clockwise, respectively. Note that since $\sigma(j,0) = \infty$, if u, v are in the bottom row of σ, they are part of a counterclockwise triple if and only if $\sigma(u) > \sigma(v)$. Extend this definition to (possibly non-standard) fillings by defining the orientation of a triple to be the orientation of the corresponding triple for the standardized filling $\zeta(\sigma)$. (So for two equal numbers, the one which occurs first in the reading word is regarded as being smaller). It is an easy exercise to show that $\mathrm{inv}(\sigma,\mu)$ is the number of counterclockwise triples. For example, for the filling in Figure 10.9, the set consisting of the inversion triples is

$$\begin{aligned}&\{\{(1,3),(1,4),(2,4)\},\{(1,2),(1,3),(2,3)\},\{(1,1),(1,2),(3,2)\},\\ &\quad\{(2,1),(2,2),(3,2)\},\{(2,1),(3,1),(2,0)\},\{(2,1),(4,1),(2,0)\}\}.\end{aligned}$$

The following theorem was conjectured by Haglund [50] and proved by Haglund, Haiman and Loehr [59], [57]. It gives a combinatorial formula for $\tilde{H}_\mu$.

Theorem 10.5.3 *For* $\mu \in \mathrm{Par}$,

$$C_\mu(X;q,t) = \tilde{H}_\mu(X;q,t). \tag{10.5.6}$$

Remark 10.5.4 Theorem 10.3.6 or (10.3.25) imply the well-known symmetry relation

$$\tilde{H}_\mu(X;q,t) = \tilde{H}_{\mu'}(X;t,q). \tag{10.5.7}$$

(This can be derived fairly easily from the three axioms in Theorem 10.5.5 below.) An interesting open question in enumerative combinatorics is to prove this symmetry combinatorially using Theorem 10.5.3. Note that in the case $\mu = (n)$ this question is equivalent to asking for a bijective proof that maj and inv have the same distribution on arbitrary multisets, which is a classical result due to Foata [30].

Proof of Theorem 10.5.3

Theorem 10.3.1, when translated into a statement about the $\tilde{H}_\mu$ using (10.3.37), gives (see [68], [57]):

Theorem 10.5.5 *The following three conditions uniquely determine a family $\tilde{H}_\mu(X; q, t)$ of symmetric functions.*

$$\tilde{H}_\mu[X(q-1); q, t] = \sum_{\rho \leq \mu'} c_{\rho,\mu}(q, t) m_\rho(X), \tag{10.5.8}$$

$$\tilde{H}_\mu[X(t-1); q, t] = \sum_{\rho \leq \mu} d_{\rho,\mu}(q, t) m_\rho(X), \tag{10.5.9}$$

$$\tilde{H}_\mu(X; q, t)|_{x_1^n} = 1. \tag{10.5.10}$$

Hence if one can show that $C_\mu(X; q, t)$ is a symmetric function and satisfies the three axioms above, it must be equal to $\tilde{H}_\mu$. The fact that $C_\mu(X; q, t)$ satisfies (10.5.10) is trivial.

Next we argue that C_μ can be written as a sum of LLT polynomials. Fix a descent set D, and let

$$G_D(X; q) = \sum_{\substack{\sigma:\mu \to \mathbb{Z}^+ \\ \mathrm{Des}(\sigma,\mu) = D}} q^{\mathrm{inv}(\sigma,\mu)} x^\sigma. \tag{10.5.11}$$

If μ has one column, then G_D is a ribbon Schur function, that is a Schur function of a skew shape containing no 2×2 square blocks of contiguous cells. More generally, $G_D(X; q)$ is an LLT product of ribbons. We illustrate how to transform a filling σ into a term $\mathbf{T}(\sigma)$ in the corresponding LLT product in Figure 10.10. Note that inversion pairs in σ are in direct correspondence with LLT inversion pairs in $\mathbf{T}(\sigma)$. Since the shape of the ribbons in $\mathbf{T}(\sigma)$ depends only on $\mathrm{Des}(\sigma, \mu)$ we have

$$C_\mu(X; q, t) = \sum_D t^L q^{-A} G_D(X; q), \tag{10.5.12}$$

where the sum is over all possible descent sets D of fillings of μ, with

$$L = \sum_{s \in D} (\mathrm{leg}(s) + 1), \qquad A = \sum_{s \in D} \mathrm{arm}(s).$$

Since LLT polynomials are symmetric functions, $C_\mu(X; q, t)$ is a symmetric function.

Since C_μ is a symmetric function, (10.5.5) combined with general results on symmetric functions implies [57], [53, Chap. 6]

$$\omega^W C_\mu[Z + W; q, t] = \sum_{\beta \in S_n} t^{\mathrm{maj}(\beta,\mu)} q^{\mathrm{inv}(\beta,\mu)} \tilde{F}_{n,\mathrm{Des}(\beta^{-1})}(Z, W), \tag{10.5.13}$$

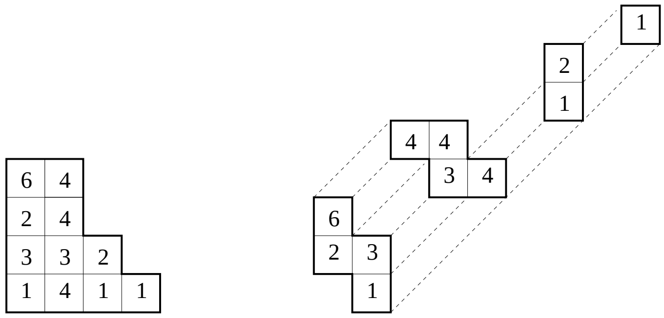

Figure 10.10 On the left, a filling σ, and on the right, the term $\mathbf{T}(\sigma)$ in the corresponding LLT product of ribbons.

where ω^W is the involution ω acting on the W-set of variables, leaving the Z-set alone, and

$$\tilde{F}_{n,D}(Z,W) = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n \\ a_i = a_{i+1} \in \mathcal{A}_+ \implies i \notin D \\ a_i = a_{i+1} \in \mathcal{A}_- \implies i \in D}} z_{a_1} z_{a_2} \cdots z_{a_n}. \tag{10.5.14}$$

Here the indices a_i range over the alphabet $\mathcal{A}_\pm = \{1, \bar{1}, 2, \bar{2}, \ldots\}$, which is the union of $\mathcal{A}_+ = \{1, 2, \ldots\}$ and $\mathcal{A}_- = \{\bar{1}, \bar{2}, \ldots\}$, and by convention $z_{\bar{a}} = w_a$. Eq (10.5.13) is known as the *superization* of C_μ, and $\tilde{F}_{n,D}(Z,W)$ the *super quasisymmetric function*. It is important to note that (10.5.13) holds no matter what ordering we take for the elements of $\mathcal{A}_\pm$; we will be working with the two orderings

$$1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n} \tag{10.5.15}$$

and

$$1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}. \tag{10.5.16}$$

Assume (10.5.15) holds. By letting $z_i = qx_i$ and $w_i = -x_i$ in (10.5.13), by an extension of Remark 10.2.8 we get an expression for $C_\mu[X_n(q-1); q, t]$ as a sum over "super fillings" $\tilde{\sigma}$ of $\mathrm{dg}(\mu)$. In [57] the authors introduce a sign-reversing involution which pairs super fillings with the same weight but opposite sign, leaving only terms whose monomial weights visibly satisfy the triangularity condition $\rho \le \mu'$ occurring on the right-hand-side of (10.5.8). Next assume (10.5.16) holds. By letting $z_i = tx_i$ and $w_i = -x_i$ in (10.5.13) we get an expression in terms of super fillings for $C_\mu[X_n(t-1); q, t]$. In [57] the authors introduce a different sign-reversing involution for this case, which after cancellation leaves only terms whose monomial weights visibly satisfy the triangularity condition $\rho \le \mu$ occurring on the right-hand-side of (10.5.9). We conclude that $C_\mu(X_n; q, t)$ satisfies all three axioms and hence must equal $\tilde{H}_\mu(X_n; q, t)$. $\square$

Remark 10.5.6 The reader will notice the similarity between the $\mathrm{inv}(\sigma, \mu)$ statistic and the dinv statistic on parking functions from Section 10.4. In fact, it was elements of the Shuffle Conjecture combined with known ways of expressing special cases of $\tilde{H}_\mu(X; q, t)$ in terms of

LLT polynomials which led the author to conjecture Theorem 10.5.3 in [50]. See [52] for a detailed description of this story.

10.6 Consequences of Theorem 10.5.3

The Cocharge Formula for Hall-Littlewood Polynomials

In this subsection we show how to derive (10.2.13), Lascoux and Schützenberger's formula for the Schur coefficients of the Hall-Littlewood polynomials, from Theorem 10.5.3. This application was first published in [59] and [57], although the exposition here is taken mainly from [52].

We require the following lemma, whose proof is due to N. Loehr and G. Warrington [115].

Lemma 10.6.1 *Let $\mu \vdash n$. Given multisets M_i, $1 \le i \le \ell(\mu)$, of positive integers with $|M_i| = \mu_i$, there is a unique filling σ with the property that the multiset of elements of σ in the ith row of μ is M_i for $1 \le i \le \ell(\mu)$, and $\mathrm{inv}(\sigma, \mu) = 0$.*

Proof Clearly the elements in the bottom row will generate no inversion triples if and only if they are in monotone nondecreasing order in the reading word. Consider the number to place in square $(1, 2)$, i.e. right above the square $(1, 1)$. Let p be the smallest element of M_2 which is strictly larger than $\sigma(1, 1)$, if it exists, and the smallest element of M_2 otherwise. Then if $\sigma(1, 2) = p$, one sees that $(1, 1)$ and $(1, 2)$ will not form any inversion triples with $(j, 2)$ for any $j > 1$. We can iterate this procedure. In square $(2, 2)$ we place the smallest element of $M_2 - \{p\}$ (the multiset obtained by removing one copy of p from M_2) which is strictly larger than $\sigma(2, 1)$, and so on, until we fill out row 2. Then we let $\sigma(1, 3)$ be the smallest element of M_3 which is strictly larger than $\sigma(1, 2)$, if it exists, and the smallest element of M_3 otherwise, etc. Each square (i, j) cannot be involved in any inversion triples with $(i, j - 1)$ and (k, j) for some $k > i$, so $\mathrm{inv}(\sigma, \mu) = 0$. For example, if $M_1 = \{1, 1, 3, 6, 7\}$, $M_2 = \{1, 2, 4, 4, 5\}$, $M_3 = \{1, 2, 3\}$ and $M_4 = \{2\}$, then the corresponding filling with no inversion triples is given in Figure 10.11. □

Given a filling σ, we construct a word $\mathrm{cword}(\sigma)$ by initializing cword to the empty string, then scanning through $\mathrm{read}(\sigma)$, from the beginning to the end, and each time we encounter a 1, adjoin the number of the row containing this 1 to the beginning of cword. After recording the row numbers of all the 1's in this fashion, we go back to the beginning of $\mathrm{read}(\sigma)$, and adjoin the row numbers of squares containing 2's to the beginning of cword. For example, if σ is the filling in Figure 10.11, then $\mathrm{cword}(\sigma) = 11222132341123$.

Assume σ is a filling with no inversion triples. We translate the statistic $\mathrm{maj}(\sigma, \mu)$ into a statistic on $\mathrm{cword}(\sigma)$. Note that $\sigma(1, 1)$ corresponds to the rightmost 1 in $\mathrm{cword}(\sigma)$ - denote this 1 by w_{11}. If $\sigma(1, 2) > \sigma(1, 1)$, $\sigma(1, 2)$ corresponds to the rightmost 2 which is left of w_{11}, otherwise it corresponds to the rightmost 2 (in $\mathrm{cword}(\sigma)$). In any case denote this 2 by w_{12}. More generally, for $i > 1$ the element in $\mathrm{cword}(\sigma)$ corresponding to $\sigma(1, i)$ is the first i encountered when travelling left from $w_{1,i-1}$, looping around and starting at the right end of $\mathrm{cword}(\sigma)$ if necessary. To find the subword $w_{21} w_{22} \cdots w_{2\mu'_2}$ corresponding to the second

2				
3	1	2		
2	4	4	1	5
1	1	3	6	7

Figure 10.11 A filling with no inversion triples.

column of σ, we do the same algorithm on the word obtained by removing the elements $w_{11}w_{12}\cdots w_{1\mu'_1}$ from cword(σ). After that we remove $w_{21}w_{22}\cdots w_{2\mu'_2}$ and apply the same process to find $w_{31}w_{32}\cdots w_{3\mu'_3}$ etc..

Clearly $\sigma(i,j) \in \mathrm{Des}(\sigma,\mu)$ if and only if w_{ij} occurs to the left of $w_{i,j-1}$ in cword(σ). Thus maj(σ,μ) is transparently equal to the statistic cocharge(cword(σ)) described in the Cocharge Algorithm from Section 10.2.

We associate a two-line array $A(\sigma)$ to a filling σ with no inversions by letting the upper row $A_1(\sigma)$ be nonincreasing with the same weight as σ, and the lower row $A_2(\sigma)$ be cword(σ). For example, to the filling in Figure 10.11 we associate the two-line array

$$\begin{array}{c} 7\,6\,5\,4\,4\,3\,3\,2\,2\,2\,1\,1\,1\,1 \\ 1\,1\,2\,2\,2\,1\,3\,2\,3\,4\,1\,1\,2\,3 \end{array} \tag{10.6.1}$$

By construction, below equal entries in the upper row the entries in the lower row are nondecreasing. Since $\tilde{H}_\mu$ is a symmetric function, we can reverse the variables, replacing x_i by x_{n-i+1} for $1 \le i \le n$, without changing the sum. This has the effect of changing $A_1(\sigma)$ into a nondecreasing word, and we end up with an ordered two-line array as in the classic RSK algorithm (see [142, Chap. 7]). We can invert this correspondence since from the two-line array we get the multiset of elements in each row of σ, which uniquely determines σ by Lemma 10.6.1. Thus (by Theorem 10.5.3)

$$\tilde{H}_\mu(x_1,x_2,\ldots,x_n;0,t) = \sum_{\sigma:\,\mathrm{inv}(\sigma,\mu)=0} x^{\mathrm{weight}(A_1(\sigma))} t^{\mathrm{cocharge}(A_2(\sigma))} \tag{10.6.2}$$

$$= \sum_{(A_1,A_2)} x^{\mathrm{weight}(A_1)} t^{\mathrm{cocharge}(A_2)}, \tag{10.6.3}$$

where the sum is over ordered two-line arrays satisfying weight$(A_2) = \mu$.

Now it is well known that for any word w of partition weight, we have cocharge(w) = cocharge(read(P_w)), where read(P_w) is the reading word of the insertion tableau P_w under the RSK algorithm [123, pp.48-49], [142, p.417]. Hence applying the RSK algorithm to (10.6.3),

$$\tilde{H}_\mu(x_1,x_2,\ldots,x_n;0,t) = \sum_{(P,Q)} x^{\mathrm{weight}(Q)} t^{\mathrm{cocharge}(\mathrm{read}(P))}, \tag{10.6.4}$$

where the sum is over all pairs (P,Q) of SSYT of the same shape with weight$(P) = \mu$. Since the number of different Q tableaux of weight ν matched to a given P tableau of shape λ is the

Kostka number $K_{\lambda,\nu}$,

$$\begin{aligned}\tilde{H}_\mu(X;0,t) &= \sum_\nu m_\nu \sum_\lambda \sum_{\substack{P\in \mathrm{SSYT}(\lambda,\mu)\\ Q\in \mathrm{SSYT}(\lambda,\nu)}} t^{\mathrm{cocharge}(\mathrm{read}(P))}\\ &= \sum_\lambda \sum_{P\in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{cocharge}(\mathrm{read}(P))} \sum_\nu m_\nu K_{\lambda,\nu}\\ &= \sum_\lambda s_\lambda \sum_{P\in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{cocharge}(\mathrm{read}(P))}. \end{aligned} \tag{10.6.5}$$

This finishes the proof of (10.2.13).

Formulas for J_μ

By (10.3.37) we have

$$J_\mu(Z;q,t) = t^{n(\mu)}\tilde{H}_\mu[Z(1-t);q,1/t] \tag{10.6.6}$$

$$= t^{n(\mu)}\tilde{H}_\mu[Zt(1/t-1);q,1/t]$$

$$= t^{n(\mu)+n}\tilde{H}_{\mu'}[Z(1/t-1);1/t,q] \tag{10.6.7}$$

using (10.5.7). Given a super filling $\tilde{\sigma}$, let $|\tilde{\sigma}|$ be the filling obtained by replacing each negative letter $\bar{i}$ by the corresponding positive letter i for all i. Say $\tilde{\sigma}$ is *nonattacking* if no two squares containing equal entries in $|\tilde{\sigma}|$ attack each other (in the sense of the paragraph above (10.5.3)). Formula (10.5.13) and the first sign-reversing involution from the proof of Theorem 10.5.3 imply

$$J_\mu(Z;q,t) = \sum_{\text{nonattacking super fillings } \tilde{\sigma} \text{ of } \mu'} z^{|\tilde{\sigma}|} q^{\mathrm{maj}(\tilde{\sigma},\mu')} t^{\mathrm{coinv}(\tilde{\sigma},\mu')}(-t)^{\mathrm{neg}(\tilde{\sigma})}, \tag{10.6.8}$$

where coinv $= n(\mu) - \mathrm{inv}$ is the number of coinversion triples, and we use the ordering $1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}$.

The following more compact form of (10.6.8) can be obtained by grouping together all the 2^n super fillings $\tilde{\sigma}$ whose absolute value equals a fixed positive filling σ.

Corollary 10.6.2 [57]

$$J_\mu(Z;q,t) = \sum_{\text{nonattacking fillings } \sigma \text{ of } \mu'} z^\sigma q^{\mathrm{maj}(\sigma,\mu')} t^{\mathrm{coinv}(\sigma,\mu')}$$

$$\times \prod_{\substack{u\in\mu'\\ \sigma(u)=\sigma(\mathrm{South}(u))}} (1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)+1}) \prod_{\substack{u\in\mu'\\ \sigma(u)\neq\sigma(\mathrm{South}(u))}} (1-t), \tag{10.6.9}$$

where coinv $= n(\mu) - \mathrm{inv}$ *is the number of coinversion triples, and each square in the bottom row is included in the last product.*

Example 10.6.3 Let $\mu = (3,3,1)$. Then for the nonattacking filling σ of μ' in Figure 10.12, we have coinversion triples $\{1,2),(2,2),(1,1)\}$, $\{(1,1),(2,1),(1,0)\}$, $\{(1,1),(3,1),(1,0)\}$ so coinv = 3. Furthermore maj = 3, squares $(1,1)$, $(1,2)$, $(2,1)$, $(2,3)$ and $(3,1)$ each contribute a $(1-t)$, square $(1,3)$ contributes a $(1-qt^2)$, and $(2,2)$ contributes a $(1-q^2t)$. Thus the term in (10.6.9) corresponding to σ is

$$x_1x_2^3x_3^2x_4q^3t^3(1-qt^2)(1-q^2t)(1-t)^5. \tag{10.6.10}$$

$$\begin{array}{|c|c|c}
\hline
2 & 4 & \multicolumn{1}{c}{} \\
\hline
2 & 3 & \multicolumn{1}{c}{} \\
\hline
1 & 3 & \multicolumn{1}{c|}{2} \\
\hline
\end{array}$$

Figure 10.12 A nonattacking filling of $(3,3,1)'$.

The (integral form) Jack polynomials $J_\mu^{(\alpha)}(Z)$ can be obtained from the Macdonald J_μ by

$$J_\mu^{(\alpha)}(Z) = \lim_{t\to 1} \frac{J_\mu(Z;t^\alpha,t)}{(1-t)^{|\mu|}}. \tag{10.6.11}$$

If we set $q = t^\alpha$ in (10.6.9) and then divide by $(1-t)^{|\mu|}$ and take the limit as $t \to 1$ we get the following result of Knop and Sahi [86].

$$J_\mu^{(\alpha)}(Z) = \sum_{\text{nonattacking fillings } \sigma \text{ of } \mu'} z^\sigma \prod_{\substack{u\in\mu' \\ \sigma(u)=\sigma(\mathrm{South}(u))}} (\alpha(\mathrm{leg}(u)+1) + \mathrm{arm}(u)+1). \tag{10.6.12}$$

Remark 10.6.4 There is another formula for J_μ [53, pp. 132–133] corresponding to the second sign-reversing involution from the proof of Theorem 10.5.3, a formula which gives the expansion of J_μ into fundamental quasisymmetric functions F_α. The terms in the formula are not as elegant as those of (10.6.9) though, and we will not describe it here.

Schur Coefficients

Since by (10.5.12) $\tilde{H}_\mu(X;q,t)$ is a positive sum of LLT polynomials, Grojnowski and Haiman's result that LLT polynomials are Schur positive [48] gives a new proof that $\tilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$. In fact, we also get a natural decomposition of $\tilde{K}_{\lambda,\mu}(q,t)$ into "LLT components". This result is completely geometric though, and it is still hoped that a purely combinatorial formula for the $\tilde{K}_{\lambda,\mu}(q,t)$ of the form (10.3.18) can be found. In this subsection we indicate how such formulas can be obtained from Theorem 10.5.3 when μ is either a hook shape or has two columns.

The Hook Case

In [50] the following equivalent form of Stembridge's formula (10.3.24) is derived from Theorem 10.5.3;

$$\tilde{K}_{\lambda,(n-k,1^k)}(q,t) = \sum_{T\in SYT(\lambda)} t^{\mathrm{maj}(T;\mu)} q^{\mathrm{comaj}(T;\,\mathrm{rev}(\mu'))}, \tag{10.6.13}$$

together with following formula of Macdonald [120, VI.8.8]: For any $0 \le d \le n$,

$$\tilde{H}_\mu[1-z;q,t] = \sum_{d=0}^{n} (-z)^d e_d[B_\mu(q,t)]. \tag{10.6.14}$$

An equivalent formulation of (10.6.14) is

$$\langle \tilde{H}_\mu, s_{n-d,1^d} \rangle = e_d[B_\mu(q,t)-1]. \tag{10.6.15}$$

Remark 10.6.5 In [53, p. 134] a solution to Remark 10.5.4 for hook shapes at the level of Schur functions is given.

The Two-Column Case

A final segment of a word is the last k letters of the word, for some k. We say a filling σ is a *Yamanouchi filling* if in any final segment of read(σ), there are at least as many i's as $i+1$'s, for all $i \ge 1$. In [57] the following result is proved.

Theorem 10.6.6 *For any partition μ with $\mu_1 \le 2$,*

$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_{\sigma \text{ Yamanouchi}} t^{\operatorname{maj}(\sigma,\mu)} q^{\operatorname{inv}(\sigma,\mu)}, \tag{10.6.16}$$

where the sum is over all Yamanouchi fillings of μ.

Other combinatorial formulas for the two-column case are known [29], [93], [150], [151] although (10.6.16) is perhaps the simplest. We also mention that Assaf and Garsia [9] have found a recursive construction which produces an explicit basis for the Garsia-Haiman modules $V(\mu)$ when μ has at most two columns or is a hook shape. The proof of Theorem 10.6.6 involves a combinatorial construction which groups together fillings which have the same maj and inv statistics. This is carried out with the aid of *crystal graphs*, which occur in the representation theory of Lie algebras. We should mention that in both (10.6.16) and (10.6.13), if we restrict the sum to those fillings with a given descent set, we get the Schur decomposition for the corresponding LLT polynomial.

If, in (10.6.16), we relax the condition that μ has at most two columns, then the equation no longer holds. It is an open problem to find a way of modifying the concept of a Yamanouchi filling so that (10.6.16) is true more generally. A specific conjecture, when μ has three columns, for the $\tilde{K}_{\lambda,\mu}(q,t)$, of the special form (10.3.18), was given in [50], and recently proved by J. Blasiak [18]. In fact, Blasiak's result applies to any LLT product of three skew-shapes, and is the most general result currently known about the combinatorics of LLT Schur coefficients. It builds on joint work of Blasiak and Fomin [19].

10.7 Nonsymmetric Macdonald Polynomials

In 1995 Macdonald [121] introduced polynomials $E_\alpha(X;q,t)$ which form a basis for the polynomial ring $\mathbb{Q}(q,t)[x_1,\ldots,x_n]$, and are natural nonsymmetric analogues of the $P_\lambda(X;q,t)$. Further development of the theory was made by Cherednik [24], Sahi [137], Knop [84], Koornwinder [87], Ion [77], and others. Both the E_α and the P_λ are special cases of much more general versions of polynomials called Macdonald-Koornwinder polynomials, which involve affine root systems, and are orthogonal with respect to a certain scalar product. See Chapter 9 of this volume for a detailed introduction to the general theory on symmetric and nonsymmetric Macdonald-Koornwinder polynomials.

In this section we will focus on the combinatorial properties of the GL_n case, where there is a special structure which allows us to assume α is a composition, i.e. $\alpha \in \mathbb{N}^n$. Given $\alpha \in \mathbb{N}^n$, let α' denote the *transpose graph* of α, consisting of the squares

$$\alpha' = \{(i, j), \quad 1 \le i \le n, \quad 1 \le j \le \alpha_i\} \tag{10.7.1}$$

Furthermore let $\mathrm{dg}(\alpha')$ denote the augmented diagram obtained by adjoining the basement row of n squares $\{(j, 0), 1 \le j \le n\}$ below α'. Given $s \in \alpha'$, we let $\mathrm{leg}(s)$ be the number of squares of α' above s and in the same column of s. Define $\mathrm{Arm}(s)$ to be the set of squares of $\mathrm{dg}(\alpha')$ which are either to the right and in the same row as s, and also in a column not taller than the column containing s, or to the left and in the row below the row containing s, and in a column strictly shorter than the column containing s. Then set $\mathrm{arm}(s) = |\,\mathrm{Arm}(s)|$. For example, for $\alpha = (1, 0, 3, 2, 3, 0, 0, 0, 0)$, the leg lengths of the squares of $(1, 0, 3, 2, 3, 0, 0, 0, 0)'$ are listed on the left in Figure 10.13 and the arm lengths on the right. Note that if α is a partition μ, the leg and arm definitions agree with those previously given for μ'.

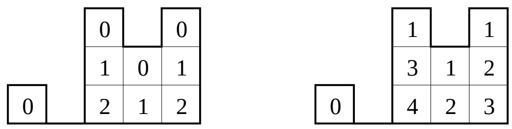

Figure 10.13 The leg lengths (on the left) and the arm lengths (on the right) for $(1, 0, 3, 2, 3, 0, 0, 0, 0)'$.

For $\alpha \in \mathbb{N}^n$, let $\mathrm{rev}(\alpha) = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$ be the composition obtained by reversing the parts of α, and set

$$\widehat{E}_\alpha(x_1, \ldots, x_n; q, t) = E_{\mathrm{rev}(\alpha)}(x_n, \ldots, x_1; 1/q, 1/t). \tag{10.7.2}$$

This modified version of the E_α is what one gets by specializing Cherednik's general theory of nonsymmetric Macdonald polynomials to GL_n root data (so the $\widehat{E}_\alpha$ is the P_α of Def. 3.9 in Chapter 9 of this volume). Formula (10.7.2) thus provides the bridge between the two natural conventions on GL_n nonsymmetric Macdonald polynomials.

Marshall [124], who made a special study of the $\widehat{E}_\alpha$, showing among other things they satisfy a version of Selberg's integral. Given two polynomials $f(x_1, \ldots, x_n; q, t)$ and $g(x_1, \ldots, x_n; q, t)$ whose coefficients depend on q, t, define a scalar product

$$\langle f, g\rangle'_{q,t} = \mathrm{CT}\, f(x_1, \ldots, x_n; q, t) g(1/x_1, \ldots, 1/x_n; 1/q, 1/t) W(x_1, \ldots, x_n; q, t). \tag{10.7.3}$$

Here CT means "take the constant term in", and

$$W(x_1,\ldots,x_n;q,t) = \prod_{1\le i<j\le n} (x_i/x_j;q)_\nu\,(qx_j/x_i;q)_\nu \tag{10.7.4}$$

where ν is a parameter such that $t = q^\nu$, and

$$(z;q)_\nu = (z;q)_\infty/(zq^\nu;q)_\infty \qquad (z;q)_\infty = \prod_{i=0}^{\infty}(1-zq^i). \tag{10.7.5}$$

Then both the $\widehat{E_\alpha}$ and the P_λ are orthogonal with respect to $\langle\,.\,,\,.\,\rangle'_{q,t}$. (This fact has an extension to general affine root systems, while it is not known whether the orthogonality of the P_λ with respect to the inner product in (10.3.2) has a version for other root systems.)

The E_α can be defined in two ways. One way, due to Macdonald [121], is to introduce a certain partial order "<" on compositions, and then show the E_α are the unique family of polynomials which are triangular in the sense that any monomials x^β that occur in E_α must satisfy $\beta \le \alpha$, and also are orthogonal with respect to $\langle,\rangle'_{q,t}$. Another way, which is more closely related to the combinatorial formulas we discuss in this section, is to use Cherednik's intertwiner relations, which give recurrence relations which uniquely define the E_α. In the GL_n case, both Knop [84] and Sahi [136] independently found a simplification in one of these two recurrence relations. The Knop-Sahi formula, (which is discussed in more detail in [60]), involves the following operators

$$\pi(\alpha_1,\ldots,\alpha_n) = (\alpha_n+1,\alpha_1,\ldots,\alpha_{n-1}), \tag{10.7.6}$$
$$\Psi f(x_1,\ldots,x_n) = x_1 f(x_2,x_3,\ldots,x_n,x_1/q), \tag{10.7.7}$$

where f is any polynomial in $\mathbb{Q}(q,t)[x_1,\ldots,x_n]$. Also, let $s_i(\alpha)$ be the composition with α_i and α_{i+1} interchanged (if $1\le i\le n-1$) and α_1 and α_n interchanged (if $i=0$).

Lemma 10.7.1 [60] *The E_α for $\alpha\in\mathbb{N}^n$ are uniquely characterized by the initial value $E_{0^n}=1$ together with the relations*

$$E_{s_i(\alpha)}(x_1,\ldots,x_n;q,t) = \left(T_i + \frac{1-t}{1-q^{l(s)+1}t^{a(s)}}\right)E_\alpha(x_1,\ldots,x_n;q,t) \tag{10.7.8}$$

(where i is such that $\alpha_i>0$ and $\alpha_{i+1}=0$) and

$$E_{\pi(\alpha)}(x_1,\ldots,x_n;q,t) = q^{\alpha_n}\Psi E_\alpha(x_1,\ldots,x_n;q,t). \tag{10.7.9}$$

In (10.7.8), the T_i, $0\le i\le n-1$ are the usual generators of the affine Hecke algebra, which satisfy the quadratic relation

$$(T_i-t)(T_i+1)=0, \tag{10.7.10}$$

together with the braid relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad T_iT_j = T_jT_i \quad \text{if} \quad |i-j|>1, \tag{10.7.11}$$

where all indices are modulo n. The T_i act on monomials in the X variables by

$$T_i x^\lambda = t x^{s_i(\lambda)} + (t-1)\frac{x^\lambda - x^{s_i(\lambda)}}{1 - x^{\gamma_i}},$$

where $x^{\gamma_i} = x_i/x_{i+1}$ for $1 \le i \le n-1$, and $x^{\gamma_0} = q x_n/x_1$.

Proof If $\alpha_1 > 0$, then $\alpha = \pi(\beta)$ where $\beta \in \mathbb{N}^N$. By induction on the sum of the parts, we can assume E_β is already determined, and apply (10.7.8). If $\alpha_1 = 0$ and $\alpha_j > 0$ for some $j > 1$, we can reduce to the case $\alpha_1 > 0$ by repeated application of (10.7.9). □

Define the "integral form" nonsymmetric Macdonald polynomials $\widehat{\mathcal{E}_\alpha}$ as

$$\widehat{\mathcal{E}_\alpha}(x_1,\dots,x_n;q,t) = \widehat{E_\alpha}(x_1,\dots,x_n;q,t)\prod_{s\in\alpha'}(1-q^{\mathrm{leg}(s)+1}t^{\mathrm{arm}(s)+1}). \tag{10.7.12}$$

Theorem 10.7.2 below describes a combinatorial formula for $\widehat{\mathcal{E}_\alpha}(X;q,t)$ which, in short, is the same as (10.6.9), using the extensions of the definitions of arm, leg, coinv, maj to composition diagrams, and changing the basement to $\sigma(j,0) = j$.

Given $\alpha \in \mathbb{N}^n$, a filling σ of α' is an assignment of integers from the set $\{1,\dots,n\}$ to the squares of α'. As before, we let the reading word read(σ) be the word obtained by reading across rows, left to right, top to bottom. The standardization of a filling σ is the filling whose reading word is the standardization of read(σ).

We say a square $s \in \alpha'$ attacks all squares to its right in its row and all squares of dg(α') to its left in the row below. Call a filling nonattacking if there are no pairs of squares (s,u) with $s \in \alpha'$, s attacks u, and $\sigma(s) = \sigma(u)$. Note that, since $\sigma(j,0) = j$, in any nonattacking filling with s of the form $(k,1)$, we must have $\sigma(s) \ge k$. Figure 10.14 gives a nonattacking filling of $(1,0,3,2,3,0,0,0,0)'$.

As before we let South(s) denote the square of dg(α') immediately below s, and let maj(σ,α') denote the sum, over all squares $s \in \alpha'$ satisfying $\sigma(s) > \sigma(\mathrm{South}(s))$, of leg($s$) + 1. A triple of α' is three squares u, v, w with $u \in \alpha'$, $v \in \mathrm{Arm}(u)$, and $w = \mathrm{South}(u)$. Note that v, w need not be in α', i.e. they could be in the basement. We determine the orientation of a triple by starting at the smallest and going in a circular motion to the next-largest and then to the largest, where if two entries of a triple have equal σ-values then the one that occurs earlier in the reading word is viewed as being smaller. We say such a triple is a coinversion triple if either v is in a column to the right of u, and u, v, w has a clockwise orientation, or v is in a column to the left of u, and u, v, w has a counterclockwise orientation. Let coinv(σ,α') denote the number of coinversion triples of σ. For example, the filling in Figure 10.14 has coinversion triples

$$\begin{aligned}&\{\{(3,2),(3,1),(4,2)\},\{(3,2),(3,1),(5,2)\},\{(3,2),(3,1),(1,1)\},\\ &\{(4,2),(4,1),(1,1)\},\{(5,1),(5,0),(1,0)\},\{(5,1),(5,0),(2,0)\},\\ &\{(5,1),(5,0),(4,0)\}\}.\end{aligned} \tag{10.7.13}$$

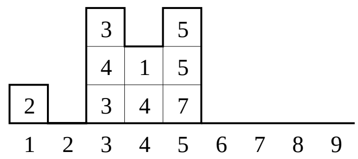

Figure 10.14 A nonattacking filling of $(1,0,3,2,3,0,0,0,0)'$.

Theorem 10.7.2 [60] *For $\alpha \in \mathbb{N}^n$, $\sum_i \alpha_i \le n$,*

$$\widehat{\mathcal{E}}_\alpha(x_1,\dots,x_n;q,t) = \sum_{\substack{\text{nonattacking fillings } \sigma \text{ of } \alpha' \\ \sigma(j,0)=j}} x^\sigma q^{\mathrm{maj}(\sigma,\alpha')} t^{\mathrm{coinv}(\sigma,\alpha')} \times \prod_{\substack{u\in\alpha' \\ \sigma(u)=\sigma(\mathrm{South}(u))}} (1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)+1}) \prod_{\substack{u\in\alpha' \\ \sigma(u)\neq\sigma(\mathrm{South}(u))}} (1-t), \quad (10.7.14)$$

where as usual $x^\sigma = \prod_{s\in\alpha'} x_{\sigma(s)}$. Equivalently,

$$\widehat{E}_\alpha(x_1,\dots,x_n;q,t) = \sum_{\substack{\text{nonattacking fillings } \sigma \text{ of } \alpha' \\ p\sigma(j,0)=j}} x^\sigma q^{\mathrm{maj}(\sigma,\alpha')} t^{\mathrm{coinv}(\sigma,\alpha')} \times \prod_{\substack{u\in\alpha' \\ \sigma(u)\neq\sigma(\mathrm{South}(u))}} \frac{(1-t)}{(1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)+1})}. \quad (10.7.15)$$

Remark 10.7.3 We can obtain corresponding versions of (10.7.14) and (10.7.15) involving the $E_\alpha(X;q,t)$ and its integral form

$$\mathcal{E}_\alpha(x_1,\dots,x_n;q,t) = E_\alpha(x_1,\dots,x_n;q,t) \prod_{s\in\mathrm{rev}(\alpha')} (1-q^{\mathrm{leg}(s)+1}t^{\mathrm{arm}(s)+1}) \quad (10.7.16)$$

by simply reversing the basement and reversing the parts of α:

$$\mathcal{E}_\alpha(x_1,\dots,x_n;q,t) = \sum_{\substack{\text{nonattacking fillings } \sigma \text{ of } (\alpha_n,\dots,\alpha_1)' \\ \sigma(j,0)=n-j+1}} x^\sigma q^{\mathrm{maj}(\sigma,\mathrm{rev}(\alpha)')} t^{\mathrm{coinv}(\sigma,\mathrm{rev}(\alpha)')} \times \prod_{\substack{u\in\mathrm{rev}(\alpha)' \\ \sigma(u)=\sigma(\mathrm{South}(u))}} (1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)+1}) \prod_{\substack{u\in\mathrm{rev}(\alpha)' \\ \sigma(u)\neq\sigma(\mathrm{South}(u))}} (1-t), \quad (10.7.17)$$

$$E_\alpha(x_1,\dots,x_n;q,t) = \sum_{\substack{\text{nonattacking fillings } \sigma \text{ of } (\alpha_n,\dots,\alpha_1)' \\ p\sigma(j,0)=n-j+1}} x^\sigma q^{\mathrm{maj}(\sigma,\mathrm{rev}(\alpha)')} t^{\mathrm{coinv}(\sigma,\mathrm{rev}(\alpha)')} \times \prod_{\substack{u\in\mathrm{rev}(\alpha)' \\ \sigma(u)\neq\sigma(\mathrm{South}(u))}} \frac{(1-t)}{(1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)+1})}. \quad (10.7.18)$$

Example 10.7.4 By (10.7.13) the nonattacking filling in Figure 10.14 has coinv = 7. There are descents at squares (1, 1), (3, 2), and (5, 1), with maj-values 1, 2, and 3, respectively. The squares (3, 1), (4, 1) and (5, 3) satisfy the condition $\sigma(u) = \sigma(\mathrm{South}(u))$ and contribute factors $(1-q^3t^5)$, $(1-q^2t^3)$ and $(1-qt^2)$, respectively. Hence the total weight associated to this filling in (10.7.14) is

$$x_1x_2x_3^2x_4^2x_5^2x_7q^6t^7(1-q^3t^5)(1-q^2t^3)(1-qt^2)(1-t)^6. \tag{10.7.19}$$

Remark 10.7.5 Let $C_\alpha(x_1,\ldots,x_n;q,t)$ denote the right-hand-side of (10.7.18). The fact that this equals $E_\alpha(x_1,\ldots,x_n;q,t)$ is proved in [60] by showing it satisfies both (10.7.8) and (10.7.9). The relation (10.7.8) actually holds term-by-term, which is proved by a simple bijective argument. To show C_α also satisfies (10.7.8) is harder, and utilizes the following result.

Lemma 10.7.6 *For any $G_1, G_2 \in \mathbb{Q}(q,t)X$, and $0 < i < n$, the following conditions are equivalent:*
(i) $G_2 = T_iG_1$;
(ii) $G_1 + G_2$ *and* $tx_{i+1}G_1 + x_iG_2$ *are symmetric in* x_i, x_{i+1}.

Lemma 10.7.6 allows one to reduce (10.7.8) to a number of technical lemmas involving super fillings and LLT polynomials.

Remark 10.7.7 The polynomials

$$\widehat{E}_\alpha(x_1,\ldots,x_n;0,0) = \widehat{\mathcal{E}}_\alpha(x_1,\ldots,x_n;0,0) \tag{10.7.20}$$

are known to equal the "standard bases" of Lascoux and Schützenberger [102], which arise in the study of Schubert varieties. These polynomials are now referred to as Demazure atoms [126], [62], since they decompose Demazure characters. Setting $q = t = 0$ in (10.7.14) gives a new combinatorial formula for the Demazure atom $\mathcal{A}_\alpha(x_1,\ldots,x_n)$, namely the sum of x^σ over all fillings σ of $\mathrm{dg}(\alpha')$ with no descents and no coinversion triples. Similarly, the special value $E_\alpha(x_1,\ldots,x_n;0,0)$ is known to equal the Demazure character, which by (10.7.18) can be expressed as the sum of x^σ over all fillings σ of $\mathrm{dg}((\alpha_n,\alpha_{n-1},\ldots,\alpha_1)')$ with no descents, no coinversion triples, and with basement $\sigma(j,0) = n-j+1$.

For any $\alpha \in \mathbb{N}^n$, let α^+ denote the partition obtained by rearranging the parts of α into nonincreasing order. It is well-known that the $P_\lambda(x_1,\ldots,x_n;q,t)$ can be expressed as a linear combination of those E_α for which $\alpha^+ = \lambda$. In terms of the $\widehat{E}_\alpha$, this identity takes the following form [124]

$$P_\lambda(X_n;q,t) = \prod_{s\in\lambda'}(1-q^{\mathrm{leg}(s)+1}t^{\mathrm{arm}(s)}) \sum_{\alpha:\alpha^+=\lambda} \frac{\widehat{E}_\alpha(x_1,\ldots,x_n;q,t)}{\prod_{s\in\alpha'}(1-q^{\mathrm{leg}(s)+1}t^{\mathrm{arm}(s)})}. \tag{10.7.21}$$

If we set $q = t = 0$ in (10.7.21), then by Remark 10.3.2 we have the identity

$$s_\lambda(X_n) = \sum_{\alpha:\alpha^+=\lambda} \mathcal{A}_\alpha(x_1,\ldots,x_n). \tag{10.7.22}$$

Sarah Mason has proved this identity bijectively by developing a generalization of the RSK algorithm [125], [126]. Haglund, Luoto, Mason and Van Willigenburg [61], [62] have used aspects of this generalized RSK algorithm to show that the product of a Schur function and a Demazure atom (character), when expanded in terms of Demazure atoms (characters), has a combinatorial interpretation, which refines the Littlewood-Richardson rule.

Remark 10.7.8 Let $\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n;q,t)$ denote the polynomial obtained by starting with the combinatorial formula (10.7.14) involving sums over nonattacking fillings, replacing the basement $(1,2,\ldots,n)$ by $(\sigma_1,\sigma_2,\ldots,\sigma_n)$, and keeping other aspects of the formula the same. Then a result first observed by Haiman, studied by Ferreira [28], and later proved by Alexandersson [3] says that if $i+1$ occurs to the left of i in the basement $(\sigma_1,\sigma_2,\ldots,\sigma_n)$, then

$$T_i\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n;q,t) = t^A\widehat{E}_\gamma^{\sigma'}(x_1,\ldots,x_n;q,t). \tag{10.7.23}$$

Here A equals one if the height of the column of $\widehat{dg}(\gamma)$ above $i+1$ in the basement is greater than or equal to the height of the column above i in the basement, and equals zero otherwise. Also, σ' is the permutation obtained by interchanging i and $i+1$ in σ. Note the quadratic relation (10.7.10) implies $T_i(T_i+1-t)=t$, or $T_i^{-1}=(T_i+1-t)/t$. By iterating (10.7.23) one can express $\widehat{E}_\gamma^\sigma$ as a certain simple power of t (depending on γ and σ) times a sequence of T_i applied to $\widehat{E}_\gamma^{n\cdots21}$, or equivalently as a certain simple power of t times T_σ^{-1} applied to $\widehat{E}_\gamma$, where T_σ^{-1} is the product of the T_i^{-1} occurring in any reduced expression for σ. Now the formula mentioned above for $\widehat{E}_\gamma^{n\cdots21}$ is the same as the formula in [60] for $E_{\gamma_n,\ldots,\gamma_1}$, which shows one can translate between the E_γ and the $\widehat{E}_\gamma$ using Hecke operators. We should mention that the Hecke algebra and the T_i have played a central role in the subject of nonsymmetric Macdonald polynomials from the outset, as in the work of Macdonald [121] and Cherednik [24]. Also, the special case $q=t=0$ of the $\widehat{E}_\gamma^\sigma$ has been studied in [64], [114], [132].

Remark 10.7.9 Let μ be a partition, and $\alpha\in\mathbb{N}^n$ with $(\alpha^+)'=\mu$, that is, a diagram obtained by permuting the columns of μ. If we let $\sigma(j,0)=\infty$, the two involutions from the proof of Theorem 10.5.3 hold for super fillings of α'. It follows that formula (10.5.4) for $\tilde{H}_\mu(X;q,t)$, and formula (10.6.9) for $J_\mu(X;q,t)$ all hold if, instead of summing over fillings of μ, we sum over fillings of α', using the definitions of arm, leg, etc. given earlier in this section.

Remark 10.7.10 Knop and Sahi [86] obtained a combinatorial formula for the nonsymmetric Jack polynomial, which is a limiting case of (10.7.14) in the same way that (10.6.12) is a limiting case of (10.6.9). In fact, it was contrasting their formula for the nonsymmetric Jack polynomials with (10.6.9) which led to (10.7.14).

10.8 The Genesis of the (q,t)-Catalan Statistics

In this section we outline the empirical steps which led the author to the discovery of the statistics on Dyck paths for the (q,t)-Catalan $C_n(q,t)$ [49]. Recall that $C_n(q,t)$, introduced by Garsia and Haiman in 1993, was originally defined as the sum of rational functions (10.3.44). Garsia and Haiman proved that $C_n(q,1)$ equals the sum of $q^{\mathrm{area}(\pi)}$ over all $\pi\in L_{n,n}^+$, and posed the problem of finding a statistic tstat to match with area so that

$$C_n(q,t) = \sum_{\pi \in L^+_{n,n}} q^{\mathrm{area}(\pi)} t^{\mathrm{tstat}(\pi)}. \tag{10.8.1}$$

By 1998 this problem and the related question of finding of finding statistics to generate $\mathrm{Hilb}(\mathrm{DH}_n; q,t)$ had become fairly well-known. At the time the author was a postdoc at MIT, and both R. Stanley and S. Billey suggested he work on the problem. The author tried several different approaches without success. A year later though, the author spent a year as a postdoc at UC San Diego, and decided to try once more. Since $C_n(q,t) = C_n(t,q)$, tstat had to have the same distribution over Dyck paths as area, but none of the other known statistics on Dyck paths equidistributed with area worked. The method the author was using was to study tables of $C_n(q,t)$, and try inventing a tstat to pair with area to match those tables.

After several attempts again met with failure, the author decided to try and look for recurrences amongst the tables, in an effort to find rules which would force certain decisions. Note that all paths in $L^+_{n,n}$ which begin with a N and then an E step are clearly in bijection with paths in $L^+_{n-1,n-1}$, and the author noticed that a copy of $t^{n-1}C_{n-1}(q,t)$ seemed to be contained in $C_n(q,t)$ in the sense that $C_n(q,t) - t^{n-1}C_{n-1}(q,t) \in \mathbb{N}[q,t]$. The author later noticed that copies of $t^{n-k}q^{\binom{k}{2}}C_{n-k}(q,t)$ were contained in $C_n(q,t)$ for all $1 \le k \le n$, and moreover

$$C_n(q,t) - \sum_{k=1}^{n} t^{n-k} q^{\binom{k}{2}} C_{n-k,n-k}(q,t) \in \mathbb{N}[q,t]. \tag{10.8.2}$$

This suggested that for any path which begins with k N steps followed by k E steps, tstat equals $n-k$ plus the value of tstat on the remaining portion of the path, viewed as an element of $L^+_{n-k,n-k}$. In particular, if to any composition α of n into positive parts we associate the "balanced" path $\pi(\alpha)$ consisting of α_1 N steps followed by α_1 E steps, then α_2 N steps followed by α_2 E steps, etc., then $\mathrm{tstat}(\pi(\alpha)) = n-\alpha_1+n-(\alpha_1+\alpha_2)+\ldots$. After a month or two of more trial and error, the author finally realized that you can associate a balanced path, called the bounce path, to any Dyck path π, via the algorithm outlined below, and that $\mathrm{tstat}(\pi)$ depends only on the bounce path.

To form the bounce path $\mathrm{bounce}(\pi)$, think of shooting a billiard ball straight north from $(0,0)$. Once the billiard ball hits the beginning of an E step of π it ricochets straight east until it hits the main diagonal $x = y$ at a point say (α_1, α_1). It then ricochets straight north and repeats the previous procedure, travelling north until it hits the beginning of an E step, then going east until it hits the diagonal, where it again ricochets north, and so on, until For an example, see the it reaches (n,n). See Figure 10.15. If the path the billiard ball takes is the balanced path which touches the diagonal at points (α_1,α_1), $(\alpha_1+\alpha_2, \alpha_1+\alpha_2)$, etc., then define the bounce statistic $\mathrm{bounce}(\pi) = (n-\alpha_1)+(n-\alpha_1-\alpha_2)+\ldots$. A short Maple program verified the conjecture that

$$C_n(q,t) = \sum_{\pi \in L^+_{n,n}} q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}(\pi)}. \tag{10.8.3}$$

for $n \le 12$.

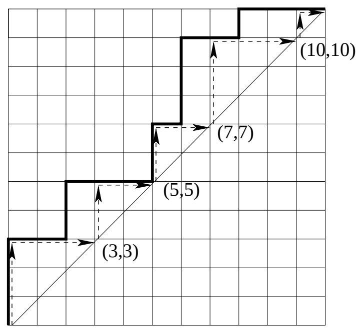

Figure 10.15 The bounce path (dotted line) for a Dyck path.

While the author and A. Garsia were trying to prove (10.8.3), M. Haiman independently found an alternate form of the conjecture involving the statistics area and dinv, as described in Section 10.4. Upon comparing the two conjectures Haiman and the author quickly realized they are equivalent, which can be proved bijectively using what we call the ζ map of a Dyck path. Given $\pi \in L^{+}_{n,n}$, let $R(\pi)$ be the parking function for π whose reading word is the reverse of the identity permutation. Construct another path $\zeta(\pi)$ by fist placing the numbers $1, 2, \ldots, n$ along the diagonal of an empty grid of squares, with i in square (i, i). Then let $\zeta(\pi)$ be the unique path with the following property: for each pair (i, j) satisfying $1 \leq i < j \leq n$, the square of the grid in the column containing i and the row containing j is below the path $\zeta(\pi)$ if an only if in $R(\pi)$, the rows containing the numbers i, j contribute an inversion to $\mathrm{dinv}(R(\pi))$. See Figure 10.16. We leave it as an exercise for the interested reader to verify that

$$\mathrm{dinv}(\pi) = \mathrm{area}(\zeta(\pi)) \quad \text{and} \quad \mathrm{area}(\pi) = \mathrm{bounce}(\zeta(\pi)). \tag{10.8.4}$$

It is still an open problem to prove $C_n(q, t) = C_n(t, q)$ bijectively, perhaps by finding a map on Dyck paths which interchanges the statistics (dinv, area), or interchanges (area, bounce). See [105] for recent work on this problem.

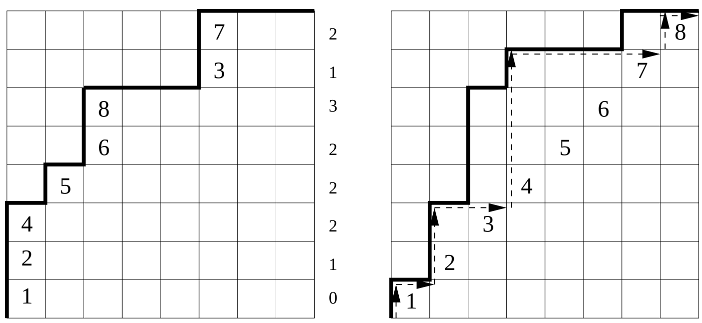

Figure 10.16 On the left, a Dyck path π, and on the right, $\zeta(\pi)$, together with its bounce path.

The bounce path arose independently in work of Andrews, Krattenthaler, Orsina, and Papi [4], who were calculating the minimal power you needed to raise an ad-nilpotent b ideal in the Lie algebra $\mathfrak{sl}(n)$ to get 0. They showed this minimal power equals the number of bounce steps of the bounce path of a certain Dyck path associated to the ideal. In a sequel to this paper the last three authors obtained versions of their results for simple Lie algebras of other type [88], in particular for type B_n. Attempts by C. Stump, the author, and others to link the bounce path of type B_n in this paper to a B_n-version of $C_n(q,t)$ defined in unpublished work of M. Haiman (which uses the Hilbert scheme) have so far been unsuccessful.

10.9 Other Directions

The Formula of Ram and Yip

In view of the combinatorial formulas (10.6.9) and (10.7.18) for the GL_n (integral form) Macdonald symmetric and nonsymmetric polynomials, a natural question to ask is whether such results exist for versions associated to other affine root systems. Progress in this direction has been made by Ram and Yip [133], who derive a closed form expression for the E_α for arbitrary affine root systems. Their formula is expressed as a sum over "alcove walks", which grew out of work of Gaussent and Littelmann [111], [112], [45], and were further developed by Lenart and Postnikov [108], [109]. They also have a corresponding formula for the symmetric P_μ for arbitrary type. When restricted to the GL_n case, their formula for P_μ in general has more terms than the Haglund-Haiman-Loehr formula, but in this case Lenart [106] has found a way of grouping terms in the Ram-Yip formula together to get a more compact formula. Rather amazingly, Lenart's more compact formula turns out to be the same as the formula for P_μ obtained as in Remark 10.7.9, with α the reverse of μ. Yip [149] used the Ram-Yip formula to obtain a "q,t-Littlewood-Richardson rule" (for arbitrary type), which expands a product of a monomial and an $E\alpha$ in terms of the E_α. The coefficients in this expansion are sums, over alcove walks, of rational functions in q,t. As corollaries she obtains expressions for the product of two E_α, or the product of a symmetric P_λ and an E_α, for arbitrary type.

Recall that Lascoux and Schützenberger's charge statistic arises when expanding the Hall-Littlewood polynomials $P_\mu(X;0,t)$ in terms of the $s_\lambda[X(1-t)]$. It is also known that

$$P_\mu(X;q,0) = \sum_\lambda K_{\lambda',\mu'}(q) s_\lambda(X). \tag{10.9.1}$$

Ion [77] has shown that for general type, the expansion of $P_\mu(X;q,0)$ in terms of Weyl characters has nonnegative coefficients (in type A, a Weyl character is a Schur function). By studying the $t=0$ case of the Ram-Yip formula, Lenart [107] has developed a version of charge for type C. Lenart and Schilling [110] have proved that this type C charge corresponds with the energy function on tensor products of Kirillov-Reshetikhin crystals.

The Probabilistic Interpretation of Diaconis and Ram

Let λ be a partition of k. Diaconis and Ram [26] introduce a Markov chain on partitions of k whose eigenfunctions are the coefficients of the (GL_n) Macdonald polynomial $P_\lambda(X_n; q, t)$, when expanded in terms of the power-sum basis $\{p_\mu(X_n)\}$. They show the stationary distribution of their Markov chain is a new two-parameter family of measures on partitions, which includes the uniform distribution on permutations and the Ewens sampling formula as special cases. Using properties of Macdonald polynomials they obtain a sharp analysis of the rate of convergence of the Markov chain.

k-Schur Functions

For any positive integer k, Lascoux, Lapointe, and Morse [100] introduced a family of symmetric functions which depend on a parameter t and reduce to Schur functions when $k = \infty$. These symmetric functions form a basis for a certain subspace of Λ. Several other conjecturally equivalent definitions of this intriguing family have been introduced; they are now commonly called k-Schur functions as in [91], denoted $\{s_\lambda^{(k)}(X; t), \lambda \in \Lambda, \lambda_1 \leq k\}$. An equivalent form of the main conjecture in [100] is that when $\tilde{H}_\mu(X; q, t)$ is expanded into the k-Schur basis with parameter q, i.e. $\{s_\lambda^{(k)}(X; q)\}$, where $k \geq \mu_1'$, the coefficients are in $\mathbf{N}[q, t]$. This and other related conjectures have sparked a large amount of research over the last ten years; see for example [92], [94], [90], [23].

Let the "bandwidth" of a LLT polynomial be the number of dotted diagonal lines which intersect the diagonal of some square in one of the skew shapes in the LLT tuple. For example, for the LLT polynomial in Figure 10.6, the bandwidth is 3, and for the tuple on the right in Figure 10.7, it is 4. It has been suggested that when expanding an LLT polynomial of bandwidth k into the k-Schur function basis $\{s_\lambda^{(k)}(X; q)\}$, the coefficients are in $\mathbf{N}[q]$. By (10.5.12), this refines the conjecture from [100] discussed in the previous paragraph.

k-Schur functions also have other remarkable properties. For example, T. Lam [89] proved a conjecture of M. Shimozono, which says that when $t = 1$ the k-Schur form the Schubert basis for the homology of the loop Grassmannian, a conjecture which was based in part on results in [95].

10.10 Recent Developments

The Work of Carlsson and Mellit

In August 2015 Carlsson and Mellit posted a preprint on the arXiv [22] which claims to prove the Compositional Shuffle Conjecture of Remark 10.4.8. Although at this time their article is still under review, their proof is generally believed by experts to be correct. As corollaries they obtain the first proof of Conjecture 10.4.1 (the combinatorial formula for $\mathrm{Hilb}(\mathrm{DH}_n; q, t)$) and more generally the first proof of the Shuffle Conjecture.

Their method of proof should have other substantial applications. Let DAHA_n denote the positive (i.e polynomial) part of the GL_n Double Affine Hecke Algebra. This involves two copies of the GL_n Affine Hecke Algebra: one containing variables $\{y_1, \ldots, y_n\}$ and corresponding T_i-operators (as in (10.7.23)) and another containing variables $\{z_1, \ldots, z_n\}$ and T_i^{-1} operators, together with a number of relations between these elements. Carlsson and Mellit introduce an abstract algebraic structure, the Double Dyck path algebra $\mathbb{A}_{q,t}$, which contains a copy of DAHA$_n$ for each $n \geq 1$, together with operators $d_+ = d_+^{(k)}$ and $d_- = d_-^{(k)}$ which map elements of DAHA$_k$ to elements of DAHA$_{k+1}$ (DAHA$_{k-1}$), respectively.

There is a wonderful action of $\mathbb{A}_{q,t}$ on elements of Λ with coefficients in $\mathbb{Q}(q,t)[y_1, \ldots, y_k]$. The T_i operators act on monomials in the y_i (as in (10.7.23) with x^λ replaced by y^λ), while the action of $d_+^{(k)}$ and $d_-^{(k)}$ is defined using plethysm. The operators d_+ and d_- are constructed so that if you start at the end of a Dyck path π and create a sequence of operators $L(\pi)$ by tracing the path backward, prepending d_+ to $L(\pi)$ for E steps and d_- to $L(\pi)$ for N steps, then $L(\pi)$ operating on the constant 1 gives a certain LLT product of single cells. Moreover, if for each EN corner of π you replace the corresponding d_-d_+ contribution to $L(\pi)$ by a factor of $(d_-d_+ - d_+d_-)/(q-1)$, then the resulting sequence $M(\pi)$, acting on the constant 1, will yield $\mathcal{F}_{\zeta^{-1}(\pi)}(X;q)$, where $\mathcal{F}$ and ζ are as in (10.4.4) and Figure 10.16.

Say we have a path π which begins with k N steps followed by an E step. Then $M(\pi)$ will begin with k d_- terms. Letting $M'(\pi)$ denote $M(\pi)$ with these k d_- terms removed, then $M'(\pi)$ applied to 1 will be a sum of symmetric functions in X with coefficients in $\mathbb{Q}(q,t)[y_1, \ldots, y_k]$. Carlsson and Mellit show that certain sums of the $M'(\pi)1$, corresponding to elements π for which $\zeta^{-1}(\pi)$ has touch points $(a_1, a_1), (a_1 + a_2, a_1 + a_2), \ldots, (n, n)$, satisfy a nice recurrence. They also show that the operator ∇ can be expressed using elements of $\mathbb{A}_{q,t}$, and then using their commutation relations prove the Compositional Shuffle Conjecture in two lines.

In a sequel to the Carlsson-Mellit paper, Mellit [127] claims to prove the Compositional Rational Shuffle Conjecture, which contains the Rational Shuffle Conjecture and Compositional Shuffle Conjecture, and hence all the conjectures from Section 10.4, as special cases. His proof starts by assuming the properties of the double Dyck path algebra developed in [22], then introduces some new ideas. In particular he relates actions of toric braids with parking functions, and exploits the known fact that DAHA$_n$ can be viewed as a quotient of the surface braid group of a torus. One question which the work of Carlsson and Mellit hasn't as of yet shed any light on is the problem of finding a combinatorial expression for the Schur coefficients of the $\mathcal{F}_\pi(X;q,t)$ of (10.4.4).

One extension of the Shuffle Conjecture which is still open is the "Delta Conjecture" of Haglund, Remmel, and Wilson [65]. For $f \in \Lambda$, let Δ_f be the linear operator defined on the $\tilde{H}_\mu$ basis via

$$\Delta_f \tilde{H}_\mu(X;q,t) = f[B_\mu(q,t)]\tilde{H}_\mu(X;q,t), \tag{10.10.1}$$

with $B_\mu(q,t)$ as in (10.3.7). The Delta Conjecture gives an elegant combinatorial formula, in terms of parking functions, for $\Delta_{e_k} e_n$, for any $0 \leq k \leq n$. For $k = n$ it reduces to the Shuffle Conjecture. To apply the method of Carlsson and Mellit we would have to express $\Delta_{e_k} e_n$ in terms of elements of $\mathbb{A}_{q,t}$, and also find new recurrences. Part of the problem is there is currently no known way to extend the Compositional Shuffle Conjecture to a Compositional Delta Conjecture, although [65] does contain such an extension for hook shapes. In a recent preprint on the arXiv Zabrocki [152] proves this special case. In another recent preprint Haglund, Rhoades, and Shimozono [66] introduce a quotient ring whose bigraded character equals the symmetric function described by the combinatorial side of the Delta Conjecture when $t = 0$. The combinatorics of this $t = 0$ case is controlled by ordered set partitions.

In another direction Sergel [140] has proved the Square Paths Conjecture of Loehr and Warrington [116], which gives a combinatorial interpretation for ∇p_n, follows from the Compositional Shuffle Conjecture. Her proof uses clever combinatorial manipulations of parking functions. Another extension of the Shuffle Conjecture due to Loehr and Warrington [117] which is still open gives a combinatorial formula, in terms of parking functions for nested Dyck paths, for the expansion of ∇s_λ into monomials, for any λ. There is also a family of conjectures connected to the combinatorics of the character of diagonal harmonics in several sets of variables under the diagonal action of S_n; see [13], [11], [12].

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