A Proof of the q,t-Catalan Positivity Conjecture

by

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Abstract.

We present here a proof that a certain rational function $C_n(q,t)$ which has come to be known as the "q,t-*Catalan*" is in fact a polynomial with positive integer coefficients. This has been an open problem since 1994. The precise form of the conjecture is given in the J. Algebraic Combin. **5** (1996), no. 3, 191–244, where it is further conjectured that $C_n(q,t)$ is the Hilbert Series of the Diagonal Harmonic Alternants in the variables $(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_n)$. Since $C_n(q,t)$ evaluates to the Catalan number at $t=q=1$, it has also been an open problem to find a pair of statistics $a(\pi),b(\pi)$ on Dyck paths π in the $n\times n$ square yielding $C_n(q,t)=\sum_\pi t^{a(\pi)}q^{b(\pi)}$. Our proof is based on a recursion for $C_n(q,t)$ suggested by a pair of statistics $a(\pi),b(\pi)$ recently proposed by J. Haglund. Thus one of the byproducts of our developments is a proof of the validity of Haglund's conjecture. It should also be noted that our arguments rely and expand on the plethystic machinery developed in Methods and Applications of Analysis, VII, **3**, (**99**), p. 363-420.

Introduction

To proceed we need to recall some definitions and notational conventions. We work with the algebra Λ of symmetric functions in a formal infinite alphabet $X=x_1,x_2,\ldots$, with coefficients in the field of rational functions $\mathbf{Q}(q,t)$. We also denote by $\Lambda_{Z[q,t]}$ the algebra of symmetric functions in X with coefficients in $Z[q,t]$. We write $\Lambda^{=d}$ for the space of symmetric functions homogeneous of degree d. The spaces $\Lambda^{\leq d}$ and $\Lambda^{>d}$ are analogously defined. We shall make extensive use here of "*plethystic*" notation. This is a notational device which simplifies manipulation of symmetric function identities. It can be easily defined and programmed in *MATHEMATICA* or *MAPLE* if we view symmetric functions as formal power series in the power symmetric functions p_k. To begin with, if $E=E[t_1,t_2,t_3,\ldots]$ is a formal Laurent series in the variables $t_1,t_2,t_3,\ldots$ (which may include the parameters q,t) we set

$$p_k[E] \;=\; E[t_1^k,t_2^k,t_3^k,\ldots]\ .$$

More generally, if a certain symmetric function F is expressed as the formal power series

$$F \;=\; Q[p_1,p_2,p_3,\ldots]$$

then we simply let

$$F[E] \;=\; Q[p_1,p_2,p_3,\ldots]\Big|_{p_k\to E[t_1^k,t_2^k,t_3^k,\ldots]}\ , \qquad \text{I.1}$$

and refer to it as "*plethystic substitution*" of E into the symmetric function F.

We make the convention that inside the plethystic brackets "[ ]", X and X_n respectively stand for $x_1+x_2+x_3+\cdots$ and $x_1+x_2+\cdots+x_n$. In particular, one sees immediately from this definition that if $f(x_1,x_2,\ldots,x_n)$ is a symmetric function then $f[X_n]=f(x_1,x_2,\ldots,x_n)$. We shall also make use of the symbols $\Omega(x)$ and $\widetilde{\Omega}(x)$ to represent the symmetric functions

$$\Omega(x) \;=\; \prod_{i\geq 1}\frac{1}{1-x_i} \quad\text{and}\quad \widetilde{\Omega}(x) \;=\; \prod_{i\geq 1}(1+x_i)\ .$$

For instance, is easily seen that in terms of $\Omega(x)$, the Cauchy, Hall-Littlewood and Macdonald kernels may be respectively be given the compact forms

$$\Omega[X_nY_m] \quad , \quad \Omega[X_nY_m(1-t)] \qquad \text{and} \qquad \Omega[X_nY_m\tfrac{1-t}{1-q}] \ .$$

This can is easily obtained by applying the defininition in I.1 to the power sum expansion

$$\Omega = \exp\Big(\sum_{k\geq 1}\frac{p_k}{k}\Big) \ .$$

Using plethystic notation we are forced to distinguish between two different minus signs. Indeed note that the definition in I.1 yields that we have

$$p_k[-X_n] \ = \ p_k[-x_1-x_2-\cdots-x_n] \ = \ -x_1^k-x_2^k-\cdots-x_n^k \ = \ -p_k[X_n] \ .$$

On the other hand, on using the ordinary meaning of the minus sign, we would obtain

$$p_k[X_n]\,\big|_{x_i\rightarrow -x_i} \ = \ (-1)^k\,p_k[X_n] \ .$$

Since both operations will necessarily occur in our formulas, we shall adopt the convention that when a certain variable has to be replaced by its negative, in the ordinary sense, then that variable will be prepended by a superscripted minus sign. Recall that the ω involution is the multiplicative extension of the operation obtained by setting

$$\omega\, p_k \ = \ (-1)^{k-1}\,p_k \ .$$

Note that the above conventions give

$$p_k[-\,^{-}X_n] \ = \ (-1)^{k-1}\,p_k[X_n] \ .$$

In particular, for any symmetric polynomial P of degree $\leq n$, we may write

$$\omega\, P[X_n] \ = \ P[-\,^{-}X_n] \ . \qquad \text{I.2}$$

Sometimes it will be convenient to use the symbol "ϵ" to represent $\,^{-}1$. The idea is that we should treat ϵ as any of the other variables in carrying out plethystic operations and only outside the plethystic bracket do we replace ϵ by -1.

A partition μ will be represented and identified with its Ferrers diagram. As customary, the partition conjugate to μ will be denoted "μ'". We shall use the French convention here and, given that the parts of μ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$, we let the corresponding Ferrers diagram have μ_i lattice cells in the i^{th} row (counting from the bottom up). It will be convenient to let $|\mu|$ and $l(\mu)$ denote respectively the sum of the parts and the number of nonzero parts of μ. In this case $|\mu| = \mu_1+\mu_2+\cdots+\mu_k$ and $l(\mu)=k$. As customary the symbol "$\mu \vdash n$" will be used to indicate that $|\mu| = n$. Following Macdonald, the *arm, leg, coarm* and *coleg* of a lattice square s are the parameters $a_\mu(s), l_\mu(s), a'_\mu(s)$ and $l'_\mu(s)$ giving the number of cells of μ that are respectively *strictly* EAST, NORTH, WEST and SOUTH of s in μ.

Here and after, for a partition $\mu=(\mu_1,\mu_2,\ldots,\mu_k)$ we set

$$n(\mu)=\sum_{i=1}^{k}(i-1)\mu_i \;=\; \sum_{s\in\mu} l'_\mu(s) \;=\; \sum_{s\in\mu} l_\mu(s) \;\;.$$

If s is a cell of μ we shall refer to the monomial $w(s)=q^{a'_\mu(s)}t^{l'_\mu(s)}$ as the *weight* of s. The sum of the weights of the cells of μ will be denoted by $B_\mu(q,t)$ and will be called the *biexponent generator of* μ. Note that we have

$$B_\mu(q,t) \;=\; \sum_{s\in\mu} q^{a'_\mu(s)}t^{l'_\mu(s)} \;=\; \sum_{i\geq 1} t^{i-1}\,\frac{1-q^{\mu_i}}{1-q} \;\;.$$

It will be also convenient to set

$$D_\mu \;=\; (1-t)(1-q)B_\mu(q,t)-1\;, \qquad \text{I.3}$$

$$T_\mu \;=\; t^{n(\mu)}q^{n(\mu')} \;=\; \prod_{s\in\mu} q^{a'_\mu(s)}t^{l'_\mu(s)}\;, \qquad \Pi_\mu(q,t) \;=\; \prod_{\substack{s\in\mu\\ s\neq(o,o)}} \left(1-q^{a'_\mu(s)}t^{l'_\mu(s)}\right) \qquad \text{I.4}$$

and finally

$$\tilde h_\mu(q,t) \;=\; \prod_{s\in\mu}\left(q^{a'_\mu(s)}-t^{l'_\mu(s)+1}\right) \;\;,\quad \tilde h'_\mu(q,t) \;=\; \prod_{s\in\mu}\left(t^{l'_\mu(s)}-q^{a'_\mu(s)+1}\right)\;. \qquad \text{I.5}$$

We shall work here with the symmetric polynomial $\tilde H_\mu[X;q,t]$ with Schur function expansion

$$\tilde H_\mu[X;q,t] \;=\; \sum_\lambda S_\lambda[X]\,\tilde K_{\lambda\mu}(q,t)\;\;, \qquad \text{I.6}$$

where the coefficients $\tilde K_{\lambda\mu}(q,t)$ are obtained from the Macdonald q,t-Kotska coefficients by setting

$$\tilde K_{\lambda\mu}(q,t) \;=\; t^{n(\mu)}\,K_{\lambda\mu}(q,1/t)\;\;.$$

Most of the properties of $\tilde H_\mu[X;,q,t]$ we will need here can be routinely derived from the corresponding properties of Macdonald's integral form $J_\mu[X;q,t]$ (†), via the formula

$$\tilde H_\mu[X;q,t]=t^{n(\mu)}J_\mu[\tfrac{X}{1-1/t}\,;q,1/t\,]\;\;. \qquad \text{I.7}$$

We shall need a number of identities satisfied by this polynomial which have been derived in previous work. To avoid unnecessary repetitions we will refer the reader to the appropriate

(†) [15] Ch. VI, (8.3)

sources whenever needed. The reader is advised to obtain copies of papers [3],[6] and [8] where most of the material we will use can be found.

The most important ingredient in the present developments is the linear operator ∇ defined, in term of the basis $\{\tilde{H}_\mu[X;q,t]\}_\mu$, by setting

$$\nabla\, \tilde{H}_\mu[X;q,t] \;=\; T_\mu\, \tilde{H}_\mu[X;q,t] \quad . \tag{I.8}$$

The reader is referred to [1], [2] and [3] for a collection of results and conjectures about ∇ that have emerged in the few years since its discovery.

Our point of departure is the basic identity

$$e_n\big[\tfrac{XY}{(1-t)(1-q)}\big] \;=\; \sum_{\mu\vdash n} \frac{\tilde{H}_\mu[X;q,t]\tilde{H}_\mu[Y;q,t]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}\,. \tag{I.9}$$

In particular, the alternate expansion

$$e_n\big[\tfrac{XY}{(1-t)(1-q)}\big] \;=\; \sum_{\mu\vdash n}(-1)^{|\mu|-l(\mu)}\frac{p_\mu[X]p_\mu[Y]}{z_\mu p_\mu[M]}\;,$$

shows that the two bases $\{\tilde{H}_\mu\}_\mu$ and $\{\tilde{H}_\mu/\tilde{h}_\mu\tilde{h}'_\mu\}_\mu$ are dual with respect to the scalar product product $\langle\;,\;\rangle_*$ defined by setting for the power basis

$$\big\langle\, p_\mu\;,\;p_\nu\big\rangle_* \;=\; (-1)^{|\mu|-l(\mu)}\chi(\mu=\nu)\, z_\mu p_\mu[M] \quad . \tag{I.10}$$

This given it may be derived from I.9 (see [6] Theorem 2.4) that we have

$$e_n[X] \;=\; \sum_{\mu\vdash n}\frac{\tilde{H}_\mu[X;q,t]M\Pi_\mu(q,t)B_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}\quad , \tag{I.11}$$

where for convenience we shall set here and after

$$M \;=\; (1-t)(1-q)\,. \tag{I.12}$$

In particular we see from I.8 and I.11 that we must have

$$\nabla e_n[X] \;=\; \sum_{\mu\vdash n}\frac{T_\mu\tilde{H}_\mu[X;q,t]M\Pi_\mu(q,t)B_\mu(q,t)}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}\quad . \tag{I.13}$$

This rational function has been conjectured in [6] to give the Frobenius characteristic of the Diagonal Harmonic polynomials in $(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_n)$. Thus the coefficients in the Schur function expansion

$$\nabla e_n[X] \;=\; \sum_{\mu\vdash n}\nabla e_n[X]\Big|_{S_\lambda} S_\lambda\;,$$

should evaluate to polynomials with positive integer coefficients.

The so called q,t-Catalan $C_n(q,t)$ was originally defined in [6] as

$$C_n(q,t) \;=\; \sum_{\mu\vdash n} \frac{T_\mu^2 M\Pi_\mu(q,t)B_\mu(q,t)}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)} \;. \qquad \text{I.14}$$

Since it can be shown (see [6]) that for $\mu \vdash n$

$$\tilde{H}_\mu[X;q,t]\Big|_{S_{1^n}} \;=\; T_\mu \;, \qquad \text{I.15}$$

we see from I.13 that we also have

$$C_n(q,t) \;=\; \nabla e_n[X]\Big|_{S_{1^n}} \;. \qquad \text{I.16}$$

Thus $C_n(q,t)$ should be the Hilbert series of the Diagonal Harmonic Alternants. For many years the only known facts about $C_n(q,t)$ remained what was shown in [6]. Namely

$$q^{\binom{n}{2}} C_n(q,1/q) \;=\; \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \qquad \text{I.17}$$

and the recursion

$$C_n(q,1) \;=\; \sum_{k=1}^{n} q^{k-1} C_{k-1}(q,1) C_{n-k}(q,1) \;. \qquad \text{I.18}$$

It was shown in [12] by high powered methods of Algebraic Geometry that $C_n(q,t)$ is a polynomial, but the positivity remained an open problem to this time. Only recently the polynomiality of $C_n(q,t)$ was obtained by elementary methods in [3] by showing that the operator ∇ acts polynomially on Schur functions. However, the recursion in I.18 suggested a very interesting combinatorial approach. To see this let us denote by $\mathcal{D}_n$ the collection of lattice paths Π in the $n\times n$ square which start at the origin $(0,0)$, proceed by NORTH and EAST steps, remaining weakly above the diagonal and stop at (n,n). Let us also define as "$area(\Pi)$" the number of lattice squares weakly below Π and strongly above the diagonal. Since we may take $C_o(q,t)\equiv 1$, from I.18 we can easily derive that we must have

$$C_n(q,1) \;=\; \sum_{\Pi\in\mathcal{D}_n} q^{area(\Pi)} \;. \qquad \text{I.19}$$

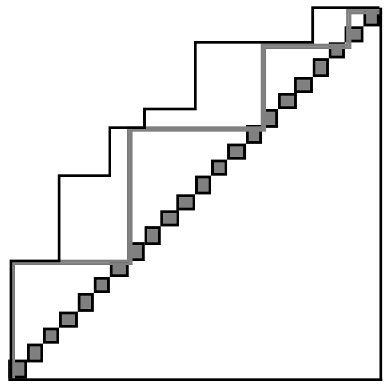
All of this led to the problem of constructing an additional statistic "$add(\Pi)$" which would extend I.19 to

$$C_n(q,t) \;=\; \sum_{\Pi\in\mathcal{D}_n} q^{area(\Pi)} t^{add(\Pi)} \;. \qquad \text{I.20}$$

We should note that since I.14 defines $C_n(q,t)$ as a symmetric function in q,t, this new statistic should also satisfy the identity

$$\sum_{\Pi\in\mathcal{D}_n} q^{area(\Pi)} \;=\; \sum_{\Pi\in\mathcal{D}_n} q^{add(\Pi)} \;. \qquad \text{I.21}$$

This problem was solved by J. Haglund in a recent paper [10] who conjectured that a possible choice for $add(\Pi)$ could be obtained as follows. Assume that a billiard ball shot straight NORTH from below the path is reflected by an EAST step of the path headed straight EAST. Moreover assume that a billiard ball headed EAST is reflected straight NORTH by the main diagonal. This given, a billiard ball shot from the origin straight NORTH will be bounced by a path $\pi \in \mathcal{D}_n$ successively into a zig-zag course (see figure below)



$$\uparrow path \rightarrow diagonal \uparrow path \rightarrow diagonal \uparrow path \cdots$$

until finally it hits the diagonal at the point (n,n). We can therefore associate to each path π the bouncing path $\beta(\pi)$ traveled by the billiard ball. Note that $\beta(\pi)$ is completely determined by the positions of its diagonal corners. If the successive diagonal corners of $\beta(\pi)$ ($(0,0)$ and (n,n) not included) are

$$(s_1, s_1)\ ,\ (s_2, s_2)\ ,\ \cdots\ ,\ (s_k, s_k)$$

with

$$0 < s_1 < s_2 < \cdots < s_k < n$$

then Haglund sets

$$maj(\beta(\Pi)) \;\;=\;\; n - s_1 + n - s_2 + \cdots + n - s_k\,.$$

Note that if we label the diagonal points

$$(1,1)\,,\ (2,2)\,,\ \ldots,\ (n-2, n-2)\,,\ (n-1, n-1)$$

successively by the integers

$$n-1\,,\ n-2\,,\ \ldots, 2\,,\ 1\,,$$

then $maj(\beta(\Pi))$ may be simply obtained by summing the labels of the diagonal corners of $\beta(\pi)$. Now the additional statistic proposed by Haglund in [10] is simply $add(\Pi) = maj(\beta(\Pi))$. More precisely it is conjectured in [10] that $C_n(q,t)$ and the polynomial

$$H_n(q,t) \;\;=\;\; \sum_{\Pi \in \mathcal{D}_n} q^{area(\Pi)} t^{maj(\beta(\Pi))} \qquad \text{I.22}$$

are one and the same. Our main result here is a proof of this conjecture.

Our point of departure is a recursion for $H_n(q,t)$ which immediately follows from the definition of the statistic $maj(\beta(\Pi))$. To this end let us denote by $\mathcal{D}_{n,s}$ the subcollection of paths in $\mathcal{D}_n$ which start with a string of s $NORTH$ steps followed by an $EAST$ step. It is then shown in [10] that the polynomials

$$H_{n,s}(q,t) \;=\; \sum_{\Pi\in\mathcal{D}_{n,s}} q^{area(\Pi)} t^{maj(\beta(\Pi))} \;. \qquad \text{I.23}$$

satisfy the recursion

$$H_{n,s}(q,t) \;=\; t^{n-s} q^{\binom{s}{2}} \sum_{r=1}^{n-s} \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q H_{n-s,r}(q,t) \;. \qquad \text{I.24}$$

Our approach in relating $H_n(q,t)$ to $C_n(q,t)$ is to use the symmetric function machinery developed in [3],[6] and [8] to show that the polynomials

$$Q_{n,s}(q,t) \;=\; t^{n-s} q^{\binom{s}{2}}\, \nabla e_{n-s}\big[X\tfrac{1-q^s}{1-q}\big]\Big|_{S_{1^{n-s}}} . \qquad \text{I.25}$$

satisfy the same recursion. More precisely, we will prove the following basic identity

Theorem I.1

For any integers $s,m \geq 1$ we have

$$\nabla e_m\big[X\tfrac{1-q^s}{1-q}\big]\Big|_{S_{1^m}} \;=\; \sum_{r=1}^{m} \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q t^{m-r} q^{\binom{r}{2}} \nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big|_{S_{1^{m-r}}} \;. \qquad \text{I.26}$$

In particular, we must necessarily have

$$H_{n,s}(q,t) \;=\; t^{n-s} q^{\binom{s}{2}}\, \nabla e_{n-s}\big[X\tfrac{1-q^s}{1-q}\big]\Big|_{S_{1^{n-s}}} . \qquad \text{I.27}$$

Note that the implication I.26 $\rightarrow$ I.27 is immediate since I.26 for $m = n-s$ gives

$$Q_{n,s}(q,t) \;=\; t^{n-s} q^{\binom{s}{2}} \sum_{r=1}^{n-s} \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q Q_{n-s,r}(q,t) \;.$$

and we can easily verify that the initial conditions

$$H_{n,n}(q,t) \;=\; q^{\binom{n}{2}} \;=\; Q_{n,n}(q,t)$$

are also satisfied.

This given, we necessarily have as corollary a proof of Haglund's conjecture:

Theorem I.2

$$\nabla e_n[X]\Big|_{S_{1^n}} \;=\; \sum_{\Pi\in\mathcal{D}_n} q^{area(\Pi)} t^{maj(\beta(\Pi))} \ .$$

To see this note that I.25 with $n{\rightarrow}n+1$ and $s{\rightarrow}1$ gives

$$H_{n+1,1}(q,t) \;=\; t^n\,\nabla\, e_n[X]\Big|_{S_{1^n}} \, ,$$

and the theorem follows since it is combinatorially evident from the definitions in I.22 and 1.23 that we also have

$$H_{n+1,1}(q,t) \;=\; t^n H_n(q,t) \;=\; t^n \sum_{\Pi\in\mathcal{D}_n} q^{area(\Pi)} t^{maj(\beta(\Pi))} \ .$$

We should mention that our efforts in establishing Theorem I.1 have yielded a number of useful summation formulas involving generalized Pieri coefficients. These formulas should have independent interest within the Theory of Symmetric Functions and we expect that they will play a role in other positivity results connected with the operator ∇ and Macdonald Polynomials. To give the flavor of these identities we will state here one that plays a crucial role in our proof of I.26. For $\nu\subseteq\mu$ a pair of partitions and $f\in\Lambda$ let us set

$$d^f_{\mu,\nu} \;=\; <f\,\tilde H_\nu \, , \, \tilde H_\mu>_* /\tilde h_\mu \tilde h'_\mu \, .$$

The $*$-duality of the two bases $\{\tilde H_\mu\}_\mu$ and $\{\tilde H_\mu/\tilde h_\mu\tilde h'_\mu\}_\mu$ then gives the expansion

$$f[X]\,\tilde H_\nu[X;q,t] \;=\; \sum_{\mu\supseteq\nu} \tilde H_\mu[X;q,t]\, d^f_{\mu,\nu} \ ,$$

where the inclusion "$\mu\supseteq\nu$" is a consequence of the Macdonald "*Pieri Rules*" (see [14] VI (6.7) and [5]). This given the following summation formula holds true in full generality.

Theorem I.3

For $A\in\Lambda^{\leq d}$ and $\nu\vdash k$ we have

$$\sum_{\substack{\mu\supseteq\nu \\ k\leq|\mu|\leq k+d}} d^A_{\mu,\nu} T_\mu \Pi_\mu \;=\; T_\nu \Pi_\nu \left(\nabla A\right)\left[MB_\nu\right] . \qquad \text{I.28}$$

The proof of I.28 is also quite interesting in its own right. It makes crucial use of the operators Δ_F defined, for a given $F\in\Lambda$, by setting for the Macdonald basis $\{\tilde H_\mu\}_\mu$

$$\Delta_F\,\tilde H_\mu \;=\; F[B_\mu]\,\tilde H_\mu \, . \qquad \text{I.29}$$

We should mention that in [3] these operators have been shown to act integrally on Schur functions, but otherwise they where studied only because they include ∇ as a special case.

This given, we expect that these operators will turn out again to be a useful tool in further work on Macdonald polynomials.

The contents of this paper are divided into five sections. In section 1 we make some preliminary observations that reduce the recursion in I.26 to a collection of m separate equations. In section 2 we recall some basic identities from [3] and [8] which are instrumental in our further developments. In section 3 we derive a number of summation formulas including Theorem I.3. In section 4 we complete the proof of Theorem I.1. In section 5 we derive representation theoretical implications of our results within the theory of Diagonal Harmonics and conclude with a few observations about promising extensions of this work.

1. The polynomial identity

Note that from the combinatorial description of the homogeneous symmetric function it follows that the expression $h_r\big[1+q+\cdots+q^{s-1}\big]$ is the generating function of partitions contained in an $(s-1)\times r$ rectangle. Consequently we may also write

$$\begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q \;=\; h_r\big[\tfrac{1-q^s}{1-q}\big] \;.$$

This permits us to rewrite the recursion in I.26 in the form

$$\nabla e_m\big[X\tfrac{1-q^s}{1-q}\big]\Big|_{S_{1^m}} \;=\; \sum_{r=1}^{m} h_r\big[\tfrac{1-q^s}{1-q}\big]\, t^{m-r}q^{\binom{r}{2}}\nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big|_{S_{1^{m-r}}} \;. \tag{1.1}$$

Now it is easily seen that both sides of this identity are polynomials in q^s. Thus the validity of 1.1 for all $s\geq 0$ is equivalent to the polynomial identity

$$\nabla e_m\big[X\tfrac{1-z}{1-q}\big]\Big|_{S_{1^m}} \;=\; \sum_{r=1}^{m} h_r\big[\tfrac{1-z}{1-q}\big]\, t^{m-r}q^{\binom{r}{2}}\nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big|_{S_{1^{m-r}}} \;. \tag{1.2}$$

This brings us to the problem of showing the equality of the polynomials on both sides of 1.2. To see what this entails we need to take a closer look at the family of polynomials $\big\{h_r\big[\tfrac{1-z}{1-q}\big]\big\}_{r\geq 0}$. To this end note that the "Cauchy" identity

$$\Omega[u\,(X-Y)] \;=\; \prod_i \frac{1-u\,y_i}{1-u\,x_i} \;=\; \sum_{m\geq 0} u^m h_m[X-Y] \tag{1.3}$$

for $X=\frac{1}{1-q}$ and $Y=\frac{z}{1-q}$ gives

$$\prod_{k\geq 0}\frac{1-uzq^k}{1-uq^k} \;=\; \Omega\big[u\,\tfrac{1-z}{1-q}\big] \;=\; \sum_{m\geq 0} u^m h_m\big[\tfrac{1-z}{1-q}\big] \;. \tag{1.4}$$

Since we obviously have

$$(1-u)\,\Omega\big[u\,\tfrac{1-z}{1-q}\big] \;=\; (1-zu)\,\Omega\big[u\,q\,\tfrac{1-z}{1-q}\big]\,,$$

equating coefficients of u^m in this equation, from the right hand side of 1.4 we derive that

$$h_m\left[\tfrac{1-z}{1-q}\right] - h_{m-1}\left[\tfrac{1-z}{1-q}\right] \;=\; q^m h_m\left[\tfrac{1-z}{1-q}\right] - z\, q^{m-1} h_{m-1}\left[\tfrac{1-z}{1-q}\right] .$$

This yields the recursion

$$h_m\left[\tfrac{1-z}{1-q}\right] \;=\; \tfrac{1-z\,q^{m-1}}{1-q^m}\, h_{m-1}\left[\tfrac{1-z}{1-q}\right]$$

and since $h_o \equiv 1$ we are finally led to the product formula

$$h_m\left[\tfrac{1-z}{1-q}\right] \;=\; \tfrac{1-z}{1-q}\,\tfrac{1-zq}{1-q^2}\cdots\tfrac{1-zq^{m-1}}{1-q^m} \;=\; \frac{(z;q)_m}{(q;q)_m} . \qquad 1.5$$

Now it develops that the polynomial basis $\{(z;q)_m\}_{m\geq 0}$ has a "Taylor" formula that may be written in the form

$$P(z) \;=\; \sum_{r\geq 0} (z;q)_r \frac{q^r}{(q;q)_r} \left(\delta_q^r P(z)\,\big|_{z=1}\right) , \qquad 1.6$$

where δ_q is the q-difference operator defined by setting

$$\delta_q P(z) \;=\; \frac{P(z) - P(z/q)}{z} . \qquad 1.7$$

Formula 1.6 is an immediate consequence of the identities

$$\delta_q^k\,(z;q)_r\,\Big|_{z=1} \;=\; \begin{cases} 0 & \text{if } k \neq r\,, \\ \frac{(q;q)_r}{q^r} & \text{if } k = r \end{cases} \qquad 1.8$$

which in turn may be obtained by iterating the simple identity

$$\delta_q(z;q)_r \;=\; \frac{(z;q)_r \,-\, (z/q;q)_r}{z} \;=\; \frac{1-q^r}{q}\;(z;q)_{r-1}\,.$$

These remarks lead us to the following beautiful conclusion

Theorem 1.1

The identity in 1.1 holds for all $s, m \geq 0$ if and only if we have

$$\delta_q^k \nabla e_m\left[X\tfrac{1-z}{1-q}\right]\Big|_{S_{1^m}}\Big|_{z=1} \;=\; \frac{t^{m-k}}{q^k}\, q^{\binom{k}{2}}\, \nabla e_{m-k}\left[X\tfrac{1-q^k}{1-q}\right]\Big|_{S_{1^{m-k}}} \qquad \forall\;\; k = 1, 2, \ldots, m\,. \qquad 1.9$$

Proof

Applying the expansion in 1.6 to the polynomial

$$P(z) = \nabla e_m\left[X\tfrac{1-z}{1-q}\right]\Big|_{S_{1^m}} \qquad 1.10$$

we obtain that

$$\nabla e_m\left[X\tfrac{1-z}{1-q}\right]\Big|_{S_{1^m}} \;=\; \sum_{r=1}^{m} (z;q)_r\, \frac{q^r}{(q,q)_r} \;\; \left(\delta_q^r \nabla e_m\left[X\tfrac{1-z}{1-q}\right]\Big|_{S_{1^m}}\Big|_{z=1}\right) \;. \qquad 1.11$$

This given, the identities in 1.9 are simply obtained from 1.5 by equating the coefficients of $(z;q)_k$ in 1.2 and 1.11 for $k=1,2,\ldots,m$.

Our next task is to eliminate the presence of z from the left hand side of 1.9. To this end note that we may write δ_q in the form

$$\delta_q \;=\; \tfrac{1}{z}\,(1-E) \tag{1.12}$$

where E is the "q-shift" operator defined by setting

$$EP(z) \;=\; P(z/q)\ . \tag{1.13}$$

It then follows that we necessarily have

$$\delta_q^k \;=\; \big(\tfrac{1}{z}\,(1-E)\big)^k \;=\; \frac{1}{z^k}(1-E)(1-qE)\cdots(1-q^{k-1}E)\,. \tag{1.14}$$

To see what this reduces to, note that the addition formula for the homogeneous symmetric functions gives that

$$\begin{aligned} h_k\big[\tfrac{1-z}{1-q}\big] &= \sum_{i=0}^{k} h_{k-i}\big[\tfrac{1}{1-q}\big]\times h_i\big[\tfrac{-z}{1-q}\big] \\ &= \sum_{i=0}^{k} h_{k-i}\big[\tfrac{1}{1-q}\big]\times(-z)^i e_i\big[\tfrac{1}{1-q}\big] \\ &= \sum_{i=0}^{k} \frac{1}{(q;q)_{k-i}}\,(-z)^i\,\frac{q^{\binom{i}{2}}}{(q;q)_i} \end{aligned}$$

and using 1.5 we may rewrite this in the form

$$(1-z)(1-zq)\cdots(1-zq^{k-1}) \;=\; \sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}_q(-1)^i\ q^{\binom{i}{2}}\ z^i\,. \tag{1.15}$$

In particular 1.14 becomes

$$\delta_q^k \;=\; \frac{1}{z^k}\ \sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}_q(-1)^i\ q^{\binom{i}{2}}\ E^i\,, \tag{1.16}$$

and so for any polynomial $P(z)$ we must have

$$\delta_q^k P(z)\Big|_{z=1} \;=\; \sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}_q(-1)^i\ q^{\binom{i}{2}}\ P(q^{-i})\,. \tag{1.17}$$

To see what becomes of the left hand side of 1.9 by means of this formula we observe that the two operations δ_q^k followed by "$\big|_{z=1}$" and ∇ followed by "$\big|_{S_{1^m}}$" can be applied in any order because the "dual" Cauchy identity

$$e_m\big[X\,\tfrac{1-z}{1-q}\big] \;=\; \sum_{\lambda\vdash m} S_\lambda\big[\tfrac{X}{1-q}\big]\,S_{\lambda'}[1-z]\ ,$$

coupled with the special evaluation

$$S_{\lambda'}[1-z] \;=\; \begin{cases} (-z)^r(1-z) & \text{if } \lambda'=(m-r,1^r)\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{1.18}$$

gives

$$e_m\big[X\tfrac{1-z}{1-q}\big] \;=\; \sum_{r=1}^{m} S_{1^{m-r},r}\big[\tfrac{X}{1-q}\big]\,(1-z)\,(-z)^{r-1}\quad . \tag{1.19}$$

This given, we see from 1.17 that we have

$$\begin{aligned} \delta_q^k e_m\big[X\tfrac{1-z}{1-q}\big]\Big|_{z=1} &= \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q (-1)^i\;\; q^{\binom{i}{2}}\; e_m\big[X\tfrac{1-q^{-i}}{1-q}\big] \\ &= \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q (-1)^{m-i}\;\; \frac{q^{\binom{i}{2}}}{q^{i\,m}}\; h_m\big[X\tfrac{1-q^{i}}{1-q}\big]\,. \end{aligned}$$

Thus we finally deduce that

$$\delta_q^k \nabla e_m\big[X\tfrac{1-z}{1-q}\big]\Big|_{z=1}\Big|_{S_{1^m}} = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q (-1)^{m-i}\;\; \frac{q^{\binom{i}{2}}}{q^{i\,m}}\; \nabla h_m\big[X\tfrac{1-q^{i}}{1-q}\big]\Big|_{S_{1^m}}. \tag{1.20}$$

To further simplify this formula and obtain another equivalent formulation of 1.9 we need to review some basic identities of the theory of Macdonald polynomials. We shall carry this out in the next section where we shall also begin to establish some special cases of 1.9.

2. Auxiliary symmetric function identities.

In this section we recall some notation and results presented in the papers [3], [6] and [8] and develop a collection of identities we will use in the proof of Theorem I.1.

We begin by recalling the following useful relations (see [6])

$$a)\quad \tilde H_\mu[1;q,t]=1 \quad,\qquad b)\quad \tilde H_\mu[X;q,t]\big|_{S_m}=1 \quad,\qquad c)\quad \tilde H_\mu[X;q,t]\big|_{S_{1^m}}=T_\mu\;. \tag{2.1}$$

We should also note the following identities

$$\tilde h_\mu(\tfrac1q,\tfrac1t) \;=\; (-1)^{|\mu|}\frac{\tilde h_\mu(q,t)}{T_\mu t^{|\mu|}}\;,\qquad \tilde h'_\mu(\tfrac1q,\tfrac1t) \;=\; (-1)^{|\mu|}\frac{\tilde h'_\mu(q,t)}{T_\mu q^{|\mu|}} \tag{2.2}$$

which are easily derived from the definitions in I.5. Morever we have ([6] Theoem 2,7)

$$T_\mu\,\omega\tilde H_\mu[X;\tfrac1q,\tfrac1t\,] \;=\; \tilde H_\mu[X;q,t\,]\;. \tag{2.3}$$

We can now immediately derive the following two useful expansions

Proposition 2.1

$$a) \quad e_n\left[\tfrac{X}{M}\right] = \sum_{\mu\vdash n} \frac{\tilde H_\mu[X;q,t]}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} \quad , \quad b) \quad h_n\left[\tfrac{X}{M}\right] = \sum_{\mu\vdash n} \frac{T_\mu \tilde H_\mu[X;q,t,]}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} \,. \qquad 2.4$$

In particular we see that

$$h_m\left[\tfrac{X}{M}\right] \;=\; \nabla e_m\left[\tfrac{X}{M}\right] \,. \qquad 2.5$$

Proof

In view of 2.1 a), formula 2.4 a) is none other than the the “Cauchy formula” in I.9 with $Y = 1$. This given, formula 2.4 b) is obtained by making the replacements $q{\rightarrow}\frac{1}{q}$ and $t{\rightarrow}\frac{1}{t}$ in 2.4 a) and using the relations in 2.2 and 2.3.

We should note that the relation between the $*$-scalar product in I.10:

$$\left\langle p_\mu \,,\, p_\nu \right\rangle_* \;=\; (-1)^{|\mu|-l(\mu)} \chi(\mu=\nu)\, z_\mu p_\mu[M]$$

and the ordinary Hall scalar product

$$\left\langle p_\mu \,,\, p_\nu \right\rangle \;=\; \chi(\mu=\nu)\, z_\mu$$

can simply be written in the form

$$\left\langle P \,,\, Q \right\rangle_* \;=\; \left\langle \omega\phi P \,,\, Q \right\rangle \quad \text{and} \quad \left\langle P \,,\, Q \right\rangle \;=\; \left\langle \omega\phi^{-1} P \,,\, Q \right\rangle_* \qquad 2.6$$

where for any symmetric polynomial P we set

$$\phi P[X] \;=\; P[MX]\,. \qquad 2.7$$

It will also be convenient to write

$$P^*[X] \;=\; \phi^{-1}P[X] \;=\; P\left[\tfrac{X}{M}\right] \,. \qquad 2.8$$

Recalling that

$$\Omega[X] = \sum_{m\geq 0} h_m[X] \quad \text{and} \quad \tilde\Omega[X] = \omega\Omega[X] = \sum_{m\geq 0} e_m[X] \qquad 2.9$$

we see from the summation

$$\tilde\Omega\left[\tfrac{XY}{M}\right] \;=\; \sum_{\mu} S_\lambda[X] S^*_{\lambda'}[Y] \qquad 2.10$$

that the pair of bases $\{S_\lambda\}_\lambda$ and $\{S^*_{\lambda'}\}_\lambda$ are dual with respect to the $*$-scalar product.

The following identity plays a crucial role in our further developments.

Theorem 2.1 (Macdonald Reciprocity)

For any pair of partitions μ and ν we have

$$\frac{\tilde{H}_\mu\big[1+uD_\lambda(q,t)\big]}{\prod_{c\in\mu}(1-u\,t^{l'_\mu(c)}q^{a'_\mu(c)})} \;=\; \frac{\tilde{H}_\lambda\big[1+uD_\mu(q,t)\big]}{\prod_{c\in\lambda}(1-u\,t^{l'_\lambda(c)}q^{a'_\lambda(c)})} \qquad 2.11$$

where we recall that we have set

$$D_\mu(q,t) \;=\; MB_\mu(q,t)-1\,. \qquad 2.12$$

In particular cancelling the common factor $(1-u)$ from both denominators and letting $u=1$, 2.11 reduces to

$$\frac{\tilde{H}_\mu\big[MB_\lambda(q,t)\big]}{\Pi_\mu(q,t)} \;=\; \frac{\tilde{H}_\lambda\big[MB_\mu(q,t)\big]}{\Pi_\lambda(q,t)}\,. \qquad 2.13$$

Formula 2.11 may be derived from an identity proved in Macdonald's original paper [13]. However, a simpler and shorter proof is given in [8].

Note that using I.9 with $Y=(1-t)(1-q^k)$ we obtain that

$$e_n\big[X\tfrac{(1-q^k)}{1-q}\big] \;=\; \sum_{\mu\vdash n}\frac{\tilde{H}_\mu[X;q,t]\tilde{H}_\mu[(1-t)(1-q^k);q,t]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}\,. \qquad 2.14$$

Now the reciprocity formula yields the following special evaluation

Proposition 2.2

$$\tilde{H}_\mu[(1-t)(1-q^k);q,t] \;=\; \Pi_\mu(q,t)\,h_k[(1-t)B_\mu(q,t)]\,(1-q^k)\,. \qquad 2.15$$

Proof

It is well known and easy to show from Macdonald's work (see [14] VI (4.8)) that

$$\tilde{H}_{(k)}[X;q,t] \;=\; (q;q)_k\,h_k\big[\tfrac{X}{1-q}\big]\,. \qquad 2.16$$

Thus 2.11 with $\lambda=(k)$ gives

$$\frac{\tilde{H}_\mu\big[MB_{(k)}(q,t)\big]}{\Pi_\mu(q,t)} \;=\; \frac{(q;q)_k\,h_k\big[\frac{(1-t)(1-q)}{1-q}B_\mu(q,t)\big]}{\Pi_{(k)}(q,t)}\,, \qquad 2.17$$

and it is easily seen that

$$B_{(k)}(q,t)=\frac{1-q^k}{1-q} \quad\text{and}\quad \Pi_{(k)}(q,t)=(q;q)_{k-1}\,.$$

Substituting this in 2.16 reduces it to

$$\frac{\tilde{H}_\mu[(1-t)(1-q^k);q,t)]}{\Pi_\mu(q,t)} \;=\; \frac{(q;q)_k\,h_k\big[(1-t)B_\mu(q,t)\big]}{(q;q)_{k-1}}$$

which is another way of writing 2.15.

To work on both sides of 1.9, and in view of 1.20, we need the following two general identities:

Proposition 2.3

$$\nabla e_m\big[X\tfrac{1-q^k}{1-q}\big]\Big|_{S_{1^m}} = (1-q^k)\sum_{\mu\vdash m}\frac{{T_\mu}^2\Pi_\mu(q,t)h_k\big[(1-t)B_\mu\big(q,t\big)\big]}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)}\,. \qquad 2.18$$

$$\nabla h_m\big[X\tfrac{1-q^k}{1-q}\big]\Big|_{S_{1^m}} = (-t)^{m-k}q^{k(m-1)}(1-q^k)\sum_{\mu\vdash m}\frac{{T_\mu}^2\Pi_\mu(q,t)e_k\big[(1-t)B_\mu\big(\tfrac1q,\tfrac1t\big)\big]}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)}\,. \qquad 2.19$$

Proof

Using 2.15 in 2.14 we obtain

$$e_m\big[X\tfrac{(1-q^k)}{1-q}\big] \;=\; \sum_{\mu\vdash m}\frac{\tilde H_\mu[X;q,t]\Pi_\mu(q,t)\,h_k[(1-t)B_\mu(q,t)\,]\,(1-q^k)}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)}\,.$$

Applying ∇ to both sides, 2.18 immediately follows from 2.1 c) upon equating coefficients of S_{1^m}. On the other hand, making the replacements $t\to\frac1t, q\to\frac1q$ and using 2.2 and 2.3 we get

$$e_m\big[X\tfrac{(1-q^{-k})}{1-q^{-1}}\big] \;=\; \sum_{\mu\vdash m}\frac{\frac{\omega\tilde H_\mu[X;q,t]}{T_\mu}\,\frac{(-1)^{m-1}\Pi_\mu(q,t)}{T_\mu}\,h_k\big[(1-\frac1t)B_\mu\big(\frac1q,\frac1t\big)\big]\,(1-q^{-k})}{\frac{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)}{T_\mu^2(qt)^m}}$$

so that applying ω to both sides and making the appropriate simplifications we finally obtain

$$\frac{1}{q^{m(k-1)}}\,h_m\big[X\tfrac{1-q^k}{1-q}\big] \;=\; (-qt)^m\sum_{\mu\vdash m}\frac{\tilde H_\mu[X;q,t]\Pi_\mu(q,t)\,h_k\big[(t-1)B_\mu\big(\frac1q,\frac1t\big)\,\big]\,(1-q^k)}{(tq)^k\tilde h_\mu(q,t)\tilde h'_\mu(q,t)}\,.$$

Since we also have

$$h_k[(t-1)B_\mu\big(\tfrac1q,\tfrac1t\big)\,] \;=\; (-1)^k e_k\big[(1-t)B_\mu\big(\tfrac1q,\tfrac1t\big)\,\big]\,,$$

we finally obtain that

$$\nabla h_m\big[X\tfrac{1-q^k}{1-q}\big] = (-t)^{m-k}q^{k(m-1)}(1-q^k)\sum_{\mu\vdash m}\frac{T_\mu\tilde H_\mu[X;q,t]\Pi_\mu(q,t)e_k\big[(1-t)B_\mu\big(\frac1q,\frac1t\big)\big]}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)}\,,$$

and 2.19 immediately follows by equating coefficients of S_{1^n}. We can thus summarize the results of this section with the following

Theorem 2.2

The recursion in I.26 will hold true for all s,m if and only if for all $1\le k\le m$ we have

$$\begin{aligned}\sum_{i=1}^{k}\left[\begin{matrix}k\\ i\end{matrix}\right]_q q^{\binom{i}{2}-i}\,(1-q^i)\,t^{m-i}\sum_{\mu\vdash m}\frac{T_\mu^2\,\Pi_\mu}{\tilde h_\mu\tilde h'_\mu}\,e_i[(1-t)\tilde B_\mu] \;&=\; \frac{t^{m-k}}{q^k}\,q^{\binom{k}{2}}\,\nabla e_{m-k}\big[X\tfrac{1-q^k}{1-q}\big]\Big|_{S_{1^{m-k}}}\\ &=\; \frac{t^{m-k}}{q^k}\,q^{\binom{k}{2}}\,(1-q^k)\sum_{\nu\vdash m-k}\frac{{T_\nu}^2\Pi_\nu(q,t)h_k\big[(1-t)B_\nu\big(q,t\big)\big]}{\tilde h_\nu(q,t)\tilde h'_\nu(q,t)}\,,\end{aligned} \qquad 2.20$$

where for convenience, here and after for any expression $E=E(q,t)$ we set $\tilde E = E\big(\frac1q,\frac1t\big)$.

Proof

Substituting the right hand side of 2.19 for $k\to i$ in 1.20 and making some simple cancellations shows that the left hand sides of 2.20 and 1.9 are identical. Thus the assertion follows from Theorem 1.1 and 2.18.

To motivate some of the preparatory work we still need to do we will take a close look at the special case $k=1$ of 2.20. With this specialization 2.20 reduces to

$$\sum_{\mu\vdash m} \frac{T_\mu^2\,\Pi_\mu}{\tilde h_\mu \tilde h'_\mu}\,\tilde B_\mu \;=\; \sum_{\nu\vdash m-1} \frac{T_\nu^2\,\Pi_\nu}{\tilde h_\nu \tilde h'_\nu}\,B_\nu\,. \qquad 2.21$$

Although an equivalent identity was already shown in [3] (Th. 4.2), we shall nevertheless outline the proof here. The reader is referred to [3] for the missing details. To begin with it was shown in [6] that we have the summation formula

$$B_\mu(q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t) \qquad 2.22$$

where "$\nu\to\mu$" is to mean that the sum is carried out over partitions ν obtained by removing one of the corners of μ and the coefficients $c_{\mu\nu}(q,t)$ are simply those appearing in the formula

$$h_1^\perp \tilde H_\mu[X;q,t] \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\tilde H_\nu[X;q,t]\,. \qquad 2.23$$

Here "$h_1^\perp$" denotes the operator which is adjoint to multiplication by h_1 with respect to the Hall scalar product. There are explicit expressions for the $c_{\mu\nu}(q,t)$ which may be easily derived from the Pieri rules for Macdonald polynomials. Using these expressions it is shown in [3] that we have

$$c_{\mu\nu}(\tfrac{1}{q},\tfrac{1}{t}) \;=\; c_{\mu\nu}(q,t)\,\frac{T_\nu}{T_\mu}\,. \qquad 2.24$$

Thus making the replacements $q\to 1/q$, $t\to 1/t$ in 2.22 we obtain that

$$B_\mu(\tfrac{1}{q},\tfrac{1}{t}) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,\frac{T_\nu}{T_\mu}\,. \qquad 2.25$$

Substituting this in the left hand side of 2.21 gives

$$\begin{aligned}\sum_{\mu\vdash m} \frac{T_\mu^2\,\Pi_\mu}{\tilde h_\mu \tilde h'_\mu}\,\tilde B_\mu \;&=\; \sum_{\mu\vdash m} \frac{T_\mu^2\,\Pi_\mu}{\tilde h_\mu \tilde h'_\mu} \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,\frac{T_\nu}{T_\mu}\\ &=\; \sum_{\nu\vdash m-1} \frac{T_\nu}{\tilde h_\nu \tilde h'_\nu} \sum_{\mu\leftarrow\nu} \frac{\tilde h_\nu \tilde h'_\nu}{\tilde h_\mu \tilde h'_\mu}\, c_{\mu\nu}(q,t)\,T_\mu \Pi_\mu \qquad 2.26\\ &=\; \sum_{\nu\vdash m-1} \frac{T_\nu}{\tilde h_\nu \tilde h'_\nu} \sum_{\mu\leftarrow\nu} d_{\mu\nu}^{e_1^*}(q,t)\,T_\mu \Pi_\mu\end{aligned}$$

where the coefficients

$$d^{e_1^*}_{\mu\nu}(q,t) \;\; = \;\; \frac{\tilde h_\nu \tilde h'_\nu}{\tilde h_\mu \tilde h'_\mu}\; c_{\mu\nu}(q,t) \tag{2.27}$$

are precisely those occuring in the Pieri rule

$$e_1^*[X]\,\tilde H_\nu[X;q,t\,] \;\; = \;\; \sum_{\mu\leftarrow\nu} d^{e_1^*}_{\mu\nu}(q,t)\,\tilde H_\mu[X;q,t\,]\,. \tag{2.28}$$

Now it is easy to see that we have

$$\Pi_\mu(q,t) \;\; = \;\; \Pi_\nu(q,t)\big(1-\tfrac{T_\mu}{T_\nu}\big)\,,$$

hence we may write

$$\sum_{\mu\leftarrow\nu} d^{e_1^*}_{\mu\nu}(q,t)\,T_\mu\Pi_\mu \;\; = \;\; T_\nu\Pi_\nu \sum_{\mu\leftarrow\nu} d^{e_1^*}_{\mu\nu}(q,t)\,\tfrac{T_\mu}{T_\nu}\big(1-\tfrac{T_\mu}{T_\nu}\big)\,. \tag{2.29}$$

Now it is shown in [3] (see formula 1.40) that we have (for $k\geq 1$)

$$\sum_{\mu\leftarrow\nu} d^{e_1^*}_{\mu\nu}(q,t)\left(\tfrac{T_\mu}{T_\nu}\right)^k \;\; = \;\; \frac{(-1)^{k-1}}{M} e_{k-1}\big[D_\nu(q,t)\big]\,,$$

and using this formula for $k=1,2$ the summation in 2.29 becomes

$$\sum_{\mu\leftarrow\nu} d^{e_1^*}_{\mu\nu}(q,t)\,T_\mu\Pi_\mu \;\; = \;\; T_\nu\Pi_\nu\, B_\nu\,. \tag{2.30}$$

Applying this identity in 2.26 proves 2.21 and the case $k=1$ of 2.20.

The next item in our agenda is to develop summation formulas analogous to 2.25 and 2.30 that will help us to reduce summations over $\mu\vdash m$ to summations over $\nu\vdash m-k$ as will be necessary to establish 2.20 for a general $k\leq m$.

The case $k=m$ of 2.20 is also quite interesting although the argument is entirely special to this case. Note that here we must show that

$$\delta_q^m\nabla e_m\big[X\tfrac{1-z}{1-q}\big]\Big|_{S_{1^m}}\Big|_{z=1} \;\; = \;\; \frac{1}{q^m}\, q^{\binom{m}{2}}\,. \tag{2.31}$$

Now it follows from the definition of Macdonald polynomials (see [14] VI 4.7) that the Schur function expansion of the integral forms may be written in the form

$$J_\mu[X;q,t\,] \;\; = \;\; \sum_{\lambda\leq\mu} S_\lambda\big[X\big]\,\xi_{\lambda,\mu}(q,t)\,.$$

Thus I.7 gives

$$\tilde H_\mu[X;q,t\,] \;\; = \;\; \sum_{\lambda\leq\mu} S_\lambda\big[\tfrac{X}{1-1/t}\big]\,\xi_{\lambda,\mu}(q,1/t)\,t^{n(\mu)},$$

or better (if $\mu \vdash m$)

$$\tilde{H}_\mu[X;q,t] \;=\; \sum_{\lambda \leq \mu} (-t)^m S_{\lambda'}\big[\tfrac{X}{1-t}\big]\, \xi_{\lambda,\mu}(q,1/t)\, t^{n(\mu)}.$$

Making the plethystic substitution $X \to (1-t)(1-z)$ we get

$$\tilde{H}_\mu[(1-t)(1-z);q,t] \;=\; \sum_{\lambda \leq \mu} (-t)^m S_{\lambda'}[1-z]\, \xi_{\lambda,\mu}(q,1/t)\, t^{n(\mu)}. \qquad 2.32$$

Since $\lambda \leq \mu$ in dominance implies $l(\lambda) \geq l(\mu)$, the equality in 2.32, using 1.18, may be rewritten as

$$\tilde{H}_\mu[(1-t)(1-z);q,t] \;=\; (-t)^{|\mu|} \sum_{r=0}^{m-l(\mu)} (-z)^r (1-z)\, \xi_{(r+1,1^{m-r-1}),\mu}(q,1/t)\, t^{n(\mu)}.$$

The only thing that matters here is that $\tilde{H}_\mu[(1-t)(1-z);q,t]$ is a polynomial in z of degree at most $m-l(\mu)+1$ and it is of degree m only when $\mu=(m)$. Since I.9 for $Y=(1-t)(1-z)$ gives

$$\delta_q^k \nabla e_m\big[X\tfrac{1-z}{1-q}\big] \;=\; \sum_{\mu \vdash m} \frac{T_\mu\, \tilde{H}_\mu[X;q,t]}{\tilde{h}_\mu \tilde{h}'_\mu}\, \delta_q^k\, \tilde{H}_\mu[(1-t)(1-z);q,t]\,, \qquad 2.33$$

we see that, for $k=m$, 2.33 reduces to

$$\delta_q^m\, \nabla e_m\big[X\tfrac{1-z}{1-q}\big]\Big|_{S_{1^m}}\Big|_{z=1} \;=\; \frac{q^{2\binom{m}{2}}}{\tilde{h}_{(m)}\tilde{h}'_{(m)}}\, \delta_q^m\, \tilde{H}_{(m)}[(1-t)(1-z);q,t]\Big|_{z=1}\,.$$

Now

$$\tilde{h}_m \;=\; \prod_{i=1}^m (q^{i-1}-t) \quad \text{and} \quad \frac{\tilde{H}_{(m)}[(1-t)(1-z);q,t]}{\tilde{h}'_\mu} = h_m\big[\tfrac{(1-t)(1-z)}{1-q}\big]\,,$$

so to verify 2.31 we need only check that

$$\frac{q^{2\binom{m}{2}}}{\prod_{i=1}^m (q^{i-1}-t)}\; \delta_q^m h_m\big[\tfrac{(1-t)(1-z)}{1-q}\big]\Big|_{z=1} \;=\; \frac{1}{q^m}\, q^{\binom{m}{2}}\,. \qquad 2.34$$

However, because of the Schur function expansion

$$h_m\big[\tfrac{(1-t)(1-z)}{1-q}\big] \;=\; \sum_{r=0}^{m-1} S_{m-r,1^r}\big[\tfrac{1-t}{1-q}\big](-z)^r(1-z)\,,$$

2.34 reduces to

$$\frac{q^{2\binom{m}{2}}}{\prod_{i=1}^m (q^{i-1}-t)}\; (-1)^m S_{1^m}\big[\tfrac{1-t}{1-q}\big]\big(1-\tfrac{1}{q^m}\big)\big(1-\tfrac{1}{q^{m-1}}\big)\cdots\big(1-\tfrac{1}{q}\big) \;=\; \frac{1}{q^m}\, q^{\binom{m}{2}}\,,$$

or better

$$S_{1^m}\big[\tfrac{1-t}{1-q}\big] \;=\; \frac{\prod_{i=1}^m (q^{i-1}-t)}{(1-q)(1-q^2)\cdots(1-q^m)}\,.$$

But this immediately follows from 1.5 upon setting $z = 1/t$.

3. Some summation formulas for general Pieri coefficients.

For a given $f \in \Lambda^{\leq d}$ and $\mu \vdash m$, let $c_{\mu\nu}^{f^\perp}(q,t)$ denote the coefficients appearing in the formula

$$f^\perp \tilde{H}_\mu[X;q,t] \;=\; \sum_{\substack{\nu\subseteq\mu \\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{f^\perp}(q,t)\,\tilde{H}_\nu[X;q,t]\,, \tag{3.1}$$

where $f^\perp$ denotes the operator Hall-adjoint to multiplication by f. We may also proceed "dually" and for $\nu \vdash k$, define the general Pieri coefficients $d_{\mu\nu}^{f}(q,t)$ via the formula

$$f[X]\tilde{H}_\nu[X;q,t] \;=\; \sum_{\substack{\mu\supseteq\nu \\ k\leq|\mu|\leq k+d}} d_{\mu\nu}^{f}(q,t)\;\tilde{H}_\mu[X;q,t]\,. \tag{3.2}$$

Now, using the $*$-duality of the bases $\{\tilde{H}_\mu\}_\mu$ and $\{\tilde{H}_\mu/\tilde{h}_\mu\tilde{h}'_\mu\}_\mu$ we then immediately derive that

$$c_{\mu\nu}^{f^\perp}(q,t) \;=\; \langle f^\perp \tilde{H}_\mu\,,\;\tilde{H}_\nu\rangle_*/\tilde{h}_\nu\tilde{h}'_\nu\;.$$

Passing to the Hall scalar product by means of 2.6 this may be rewritten as

$$\begin{aligned} c_{\mu\nu}^{f^\perp}(q,t)\;\tilde{h}_\nu\tilde{h}'_\nu \;&=\; \langle f^\perp\tilde{H}_\mu\,,\;\omega\,\phi\,\tilde{H}_\nu\rangle \\ &=\; \langle \tilde{H}_\mu\,,\;f\,\omega\,\phi\,\tilde{H}_\nu\rangle \\ &=\; \langle \tilde{H}_\mu\,,\;\omega\,\phi\,\big(\omega f^*\,\tilde{H}_\nu\big)\rangle \\ &=\; \langle \tilde{H}_\mu\,,\;\omega f^*\,\tilde{H}_\nu\rangle_*\;. \end{aligned} \tag{3.3}$$

Since by the definition in 3.2 we should have

$$(\omega f^*)\tilde{H}_\nu[X;q,t] \;=\; \sum_{\substack{\mu\supseteq\nu \\ k\leq|\mu|\leq k+d}} d_{\mu\nu}^{\omega f^*}(q,t)\;\tilde{H}_\mu[X;q,t] \tag{3.4}$$

we can deduce from 3.3 the important identity

$$c_{\mu\nu}^{f^\perp}(q,t)\;\tilde{h}_\nu\tilde{h}'_\nu \;=\; d_{\mu\nu}^{\omega f^*}(q,t)\,\tilde{h}_\mu\tilde{h}'_\mu\;\;, \tag{3.5}$$

which may be viewed as a general form of 2.27.

Before we can state and prove our first summation formula we must recall what is perhaps one of the most striking results in the theory of Macdonald polynomials. To this end, let us recall that the "*translation by one*" operator $\mathcal{T}_1$ is defined by setting for any $P \in \Lambda$

$$\mathcal{T}_1\,P[X] \;=\; P[X+1]\;. \tag{3.6}$$

Note that since $S_\nu[1] \neq 0$ only when $\nu = (k)$ we have the Schur function expansion

$$S_\lambda[X+1] \;=\; \sum_{\nu\subseteq\lambda} S_{\mu/\nu}[X]\,S_\nu[1\,] \;=\; \sum_{k\geq 0} S_{\mu/(k)}[X]\,, \tag{3.7}$$

which may also be written in the form

$$\mathcal{T}_1 S_\lambda \;=\; \sum_{k\geq 0} h_k^\perp\, S_\lambda \,.$$

This gives

$$\mathcal{T}_1 \;=\; \sum_{k\geq 0} h_k^\perp \,. \qquad 3.8$$

Similarly, we show that the translation by $-\epsilon$ defined by setting :

$$\mathcal{T}_{-\epsilon} \;=\; P[X-\epsilon\,] \qquad 3.9$$

has the expansion

$$\mathcal{T}_{-\epsilon} \;=\; \sum_{k\geq 0} e_k^\perp \,. \qquad 3.10$$

This given, the following remarkable result was proved in [8]

Theorem 3.1

Let

$$\mathbf{\Pi} \;=\; \nabla^{-1}\mathcal{T}_{-\epsilon} \qquad 3.11$$

and for a given symmetric function P set

$$\mathbf{\Pi}_P \;=\; \mathbf{\Pi}P \;=\; \nabla^{-1}P[X-\epsilon\,]\,.$$

Then we have

$$\big\langle P\;,\;\tilde H_\mu[X+1;q,t\,]\big\rangle_* \;=\; \mathbf{\Pi}_P\big[D_\mu(q,t)\,\big]\,. \qquad 3.12$$

This identity yields our first summation formula.

Theorem 3.2

For $f\in\Lambda^{\leq d}$ and $\mu\vdash m$ we have

$$\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{f\perp}(q,t) \;=\; F[D_\mu] \qquad 3.13$$

with

$$F[X] \;=\; \nabla^{-1}\big((\omega f)\big[\tfrac{X-\epsilon}{M}\big]\big)\,. \qquad 3.14$$

Proof

This is an immediate corollary of Theorem 3.1. Indeed 3.12 for $P=\omega f^*$ gives

$$\begin{aligned}\mathbf{\Pi}_{\omega f^*}\big[D_\mu(q,t)\big] &= \big\langle \omega f^*\,,\,\mathcal{T}_1\tilde H_\mu\big\rangle_* = \big\langle f\,,\,\mathcal{T}_1\tilde H_\mu\big\rangle\\ &= \big\langle 1\,,\,f^\perp\mathcal{T}_1\tilde H_\mu\big\rangle\\ &= \big\langle 1\,,\,\mathcal{T}_1 f^\perp\tilde H_\mu\big\rangle\\ \text{(using 3.1)} &= \sum_{\substack{\nu\subseteq\mu\\ m-d\le|\nu|\le m}} c_{\mu\nu}^{f^\perp}(q,t)\ \big\langle 1\,,\,\mathcal{T}_1\tilde H_\nu\big\rangle\,.\end{aligned} \qquad 3.15$$

Note that 3.8 and 2.1 b) give

$$\big\langle 1\,,\,\mathcal{T}_1\tilde H_\nu\big\rangle = \sum_{k\ge 0}\big\langle 1\,,\,h_k^\perp\tilde H_\nu\big\rangle = \sum_{k\ge 0}\big\langle h_k\,,\,\tilde H_\nu\big\rangle = 1\,,$$

and using this in 3.15 we obtain 3.13 with 3.14.

Remark 3.1

We should note that the sequence of equalities

$$\begin{aligned}\big\langle \omega(f^\perp P)\,,\,Q\big\rangle &= \big\langle (f^\perp P)\,,\,\omega Q\big\rangle = \big\langle P\,,\,f\omega Q\big\rangle\\ &= \big\langle \omega P\,,\,(\omega f)Q\big\rangle = \big\langle (\omega f)^\perp\omega P\,,\,Q\big\rangle\end{aligned}$$

valid for any $f,P,Q\in\Lambda$, shows that for any $f,P\in\Lambda$ we have

$$\omega\,(f^\perp P) \;=\; (\omega f)^\perp\omega P\,.$$

Proposition 3.1

For

$$f[X;q,t] \;=\; \sum_\lambda c_\lambda(q,t)\,S_\lambda[X]$$

set

$$\tilde f[X;q,t] \;=\; \sum_\lambda c_\lambda\big(\tfrac{1}{q},\tfrac{1}{q}\big)\,S_\lambda[X]\,.$$

This given, we have

$$c_{\mu\nu}^{f^\perp}\big(\tfrac1q,\tfrac1t\big) \;=\; \frac{T_\nu}{T_\mu}\,c_{\mu\nu}^{(\omega\tilde f)^\perp}(q,t)\,. \qquad 3.16$$

Proof

By definition for $f\in\Lambda^{\leq d}$ and $\mu\vdash m$ we have

$$f^{\perp}\,\tilde H_\mu=\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{f^{\perp}}(q,t)\,\tilde H_\nu\ .$$

Applying ω to both sides and using the above Remark we get

$$(\omega f)^{\perp}\,\omega\tilde H_\mu=\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{f^{\perp}}(q,t)\,\omega\tilde H_\nu\ ,$$

and making the substitutions $q\to\frac{1}{q}$, $t\to\frac{1}{t}$ yields

$$(\omega\tilde f)^{\perp}\,\omega\tilde H_\mu\big[X;\tfrac{1}{q},\tfrac{1}{t}\big]=\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{f^{\perp}}(\tfrac{1}{q},\tfrac{1}{t})\,\omega\tilde H_\nu\big[X;\tfrac{1}{q},\tfrac{1}{t}\big]\ .$$

Multiplying both sides by T_μ and using 2.3 we finally obtain that

$$(\omega\tilde f)^{\perp}\,\tilde H_\mu[X;q,t\,]=\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{f^{\perp}}(\tfrac{1}{q},\tfrac{1}{t})\,\frac{T_\mu}{T_\nu}\ \tilde H_\nu[X;q,t\,]\ . \tag{3.17}$$

On the other hand, again by definition we should have

$$(\omega\tilde f)^{\perp}\,\tilde H_\mu[X;q,t\,]=\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{(\omega\tilde f)^{\perp}}(q,t)\,\tilde H_\nu[X;q,t\,]\ . \tag{3.18}$$

Comparing 3.17 and 3.18 gives 3.16 as asserted.

This result has the following immediate Corollary.

Proposition 3.2

If for some $\mu\vdash m$ and $f\in\Lambda^{\leq d}$ we have

$$\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{f^{\perp}}(q,t)\quad=\quad F[D_\mu]\,, \tag{3.19}$$

then

$$\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{(\omega\tilde f)^{\perp}}(q,t)\,T_\nu\quad=\quad T_\mu\,\tilde F[\tilde D_\mu]\,. \tag{3.20}$$

Proof

Making the substitutions $q{\rightarrow}\frac{1}{q}$, $t{\rightarrow}\frac{1}{t}$ on both sides of 3.19 and using 3.16 we get 3.20 since

$$F[D_\mu]\Big|_{\substack{q\rightarrow 1/q\\ t\rightarrow 1/t}} \;=\; \tilde F[\tilde D_\mu]\,.$$

Proposition 3.3

If we define the operator $\tilde\nabla$ by setting for the Schur function basis

$$\tilde\nabla\, S_\lambda \;=\; (\nabla S_\lambda)\Big|_{\substack{q\rightarrow 1/q\\ t\rightarrow 1/t}}$$

then

$$\tilde\nabla \;=\; \omega\,\nabla^{-1}\omega\,. \qquad 3.21$$

Proof

From the definition in I.8 we derive that

$$\tilde\nabla\, \tilde H_\mu[X;\tfrac{1}{q},\tfrac{1}{t}] \;=\; \nabla \tilde H_\mu\Big|_{\substack{q\rightarrow 1/q\\ t\rightarrow 1/t}} \;=\; \frac{1}{T_\mu}\,\tilde H_\mu[X;\tfrac{1}{q},\tfrac{1}{t}\,]\,,$$

and this is equivalent to

$$\omega\tilde\nabla\,\omega\;\omega\tilde H_\mu[X;\tfrac{1}{q},\tfrac{1}{t}] \;=\; \frac{1}{T_\mu}\,\omega\,\tilde H_\mu[X;\tfrac{1}{q},\tfrac{1}{t}\,]\,.$$

Multiplying both sides by T_μ and using 2.3 proves 3.21.

We are now ready to establish the following summation formula

Theorem 3.3

For $g\in\Lambda^{\leq d}$ and $\mu\vdash m$ we have

$$\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{(\omega g)^\perp}(q,t)\,T_\nu \;=\; T_\mu\, G[\widetilde D_\mu] \qquad 3.22$$

with

$$G[X] \;=\; \omega\nabla\Big(g\big[\tfrac{X+1}{\widetilde M}\big]\Big)\,. \qquad 3.23$$

Proof

Using Proposition 3.2 we see that Theorem 3.2 implies 3.20 that is

$$\sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c_{\mu\nu}^{(\omega\widetilde f\,)^\perp}(q,t)\,T_\nu \;=\; T_\mu\,\widetilde F[\widetilde D_\mu]\,. \qquad 3.24$$

Making the replacement $\widetilde{f}\to g$ this gives 3.22 with $G=\widetilde{F}\big|_{\tilde f\to g}$ and F given by 3.14. In other words

$$\begin{aligned} G &= \widetilde{\nabla}^{-1}\big((\omega\widetilde{f})\big[\tfrac{X-\epsilon}{\widetilde{M}}\big]\big|_{\widetilde{f}\to g} \\ &= \widetilde{\nabla}^{-1}\big((\omega g)\big[\tfrac{X-\epsilon}{\widetilde{M}}\big]\big) \\ \text{(using 3.21)}\ \to\ &= \omega\nabla\omega\big((\omega g)\big[\tfrac{X-\epsilon}{\widetilde{M}}\big]\big) \\ \text{(using I.2)}\ \to\ &= \omega\nabla\big((\omega g)\big[\tfrac{-\epsilon X-\epsilon}{\widetilde{M}}\big]\big) \\ \text{(again using I.2)}\ \to\ &= \omega\nabla\big(g\big[\tfrac{X+1}{\widetilde{M}}\big]\big) \end{aligned}$$

as desired.

Our final effort in this section is to establish formula I.28:

Proof of Theorem I.3

By linearity we need only prove the identity

$$\sum_{\substack{\mu\supseteq\nu\\ |\mu|=k+d}} d^A_{\mu,\nu}T_\mu\Pi_\mu \;=\; T_\nu\Pi_\nu\,\big(\nabla A\big)\big[MB_\nu\big] \qquad 3.25$$

for A homogeneous of degree d and ν a partition of k. To begin, the case $d=0$ is trivial since $\nabla 1=1$ and 3.25 reduces to $T_\nu\Pi_\nu=T_\nu\Pi_\nu$.

For $d\geq 1$ our point of departure is the definition in 3.2 with $f\to A$ and $X\to MB_\lambda$ for some arbitrary partition λ. We thus have

$$\sum_{\substack{\mu\supseteq\nu\\ |\mu|=k+d}} d^A_{\mu,\nu}\tilde H_\mu[MB_\lambda;q,t] \;=\; A\big[MB_\lambda\big]\tilde H_\nu[MB_\lambda;q,t]\ ,$$

and a double use of reciprocity (eq. 2.11) then gives

$$\sum_{\substack{\mu\supseteq\nu\\ |\mu|=k+d}} d^A_{\mu,\nu}\,\Pi_\mu\,\frac{\tilde H_\lambda[MB_\mu;q,t]}{\Pi_\lambda} \;=\; A\big[MB_\lambda\big]\,\Pi_\nu\,\frac{\tilde H_\lambda[MB_\nu;q,t]}{\Pi_\lambda}\ . \qquad 3.26$$

This identity may be given a more stricking form by means of the operators Δ_F mentioned in the introduction. Indeed, by chosing $F=\phi A$ in I.29, and cancelling the common factor Π_λ, formula 3.26 may be rewritten as

$$\sum_{\substack{\mu\supseteq\nu\\ |\mu|=k+d}} d^A_{\mu,\nu}\,\Pi_\mu\,\tilde H_\lambda[MB_\mu;q,t] \;=\; \Pi_\nu\,\big(\Delta_{\phi A}\tilde H_\lambda\big)[MB_\nu;q,t]\ ,$$

which, by linearity implies that for all $G\in\Lambda$:

$$\sum_{\substack{\mu\supseteq\nu\\ |\mu|=k+d}} d^A_{\mu,\nu}\,\Pi_\mu\,G[MB_\mu] \;=\; \Pi_\nu\,\big(\Delta_{\phi A}G\big)[MB_\nu]\ . \qquad 3.27$$

Comparing the left hand sides of 3.25 and 3.27, we see that, to prove 3.25 when $\mu \vdash m$ and $m = k + d$ we should take $G[X] = e_m[\frac{X}{M}] = e_m^*$. With this choice 3.27 becomes

$$\sum_{\substack{\mu \supseteq \nu \\ |\mu|=k+d}} d^A_{\mu,\nu}\, \Pi_\mu\, T_\mu \;=\; \Pi_\nu \left(\Delta_{\phi A} e_m^*\right)[MB_\nu] \,. \tag{3.28}$$

We are thus left with the task of showing that for any $A \in \Lambda^{=d}$, $\nu \vdash k$ and $m = k + d$ we have

$$\left(\Delta_{\phi A} e_m^*\right)[MB_\nu] \;=\; T_\nu \left(\nabla A\right)[MB_\nu] \,. \tag{3.29}$$

To begin with, note that from 2.4 a) and the definition of $\Delta_{\phi A}$ we get that

$$\Delta_{\phi A} e_m^* \;=\; \sum_{\mu \vdash m} \frac{A[MB_\mu]\tilde{H}_\mu}{\tilde{h}_\mu \tilde{h}'_\mu} \,. \tag{3.30}$$

To evaluate this we seek an $f \in \Lambda^{\leq d}$ which gives

$$A[MB_\mu] \;=\; \sum_{\substack{\alpha \subseteq \mu \\ m-d \leq |\alpha| \leq m}} c^{f^\perp}_{\mu\alpha} \,. \tag{3.31}$$

Note that Theorem 3.2 implies f should be chosen so that

$$A[MB_\mu] \;=\; F[D_\mu] \tag{3.32}$$

with

$$F \;=\; \nabla^{-1} \mathcal{T}_{-\epsilon}\, \omega f^* \,. \tag{3.33}$$

Now to assure 3.32 we must take

$$F[X] \;=\; A[X+1] \;=\; \mathcal{T}_1 A \,.$$

With this choice 3.33 becomes

$$\mathcal{T}_1 A \;=\; \nabla^{-1} \mathcal{T}_{-\epsilon}\, \omega f^* \,,$$

yielding

$$\omega f^* \;=\; \mathcal{T}_\epsilon \nabla \mathcal{T}_1 A \,. \tag{3.34}$$

Keeping this choice in mind, let us now substitute 3.31 in 3.30 and get

$$\Delta_{\phi A} e_m^* \;=\; \sum_{\mu \vdash m} \frac{\tilde{H}_\mu}{\tilde{h}_\mu \tilde{h}'_\mu} \sum_{\substack{\alpha \subseteq \mu \\ m-d \leq |\alpha| \leq m}} c^{f^\perp}_{\mu\alpha} \,.$$

A change of order of summation combined with 3.5 then gives

$$\Delta_{\phi A} e_m^* \;=\; \sum_{r=0}^{d} \sum_{\alpha \vdash m-r} \frac{1}{\tilde{h}_\alpha \tilde{h}'_\alpha} \sum_{\substack{\mu \supseteq \alpha \\ \mu \vdash m}} \tilde{H}_\mu d^{\omega f^*}_{\mu\alpha} \,. \tag{3.35}$$

Now recall that

$$d_{\mu\alpha}^{\omega f^*} \;=\; \big\langle \tilde H_\mu \,,\, \omega f^* \tilde H_\alpha \big\rangle_* / \tilde h_\mu \tilde h'_\mu \,, \tag{3.36}$$

and taking into account that $\mu \vdash m$ and $\alpha \vdash m-r$, we see that only the homogeneous component of ωf^* of degree degree r will contribute to the right hand side of 3.36. Denoting this component by "ωf_r^*", 3.36 becomes

$$d_{\mu\alpha}^{\omega f^*} \;=\; \big\langle \tilde H_\mu \,,\, \omega f^* \big|_r \tilde H_\alpha \big\rangle_* / \tilde h_\mu \tilde h'_\mu \;=\; d_{\mu\alpha}^{\omega f_r^*} \,. \tag{3.37}$$

This allows us to rewrite 3.35 in the form

$$\begin{aligned} \Delta_{\phi A} e_m^* \;&=\; \sum_{r=0}^{d} \sum_{\alpha \vdash m-r} \frac{1}{\tilde h_\alpha \tilde h'_\alpha} \sum_{\substack{\mu \supseteq \alpha \\ \mu \vdash m}} \tilde H_\mu d_{\mu\alpha}^{\omega f_r^*} \\ &=\; \sum_{r=0}^{d} \sum_{\alpha \vdash m-r} \frac{1}{\tilde h_\alpha \tilde h'_\alpha}\, \omega f_r^* \; \tilde H_\alpha \\ &=\; \sum_{r=0}^{d} \omega f_r^* \sum_{\alpha \vdash m-r} \frac{1}{\tilde h_\alpha \tilde h'_\alpha}\, \tilde H_\alpha \\ &=\; \sum_{r=0}^{d} \omega f_r^* \; e_{m-r}\big[\tfrac{X}{M}\big] \,. \end{aligned}$$

Thus

$$\big(\Delta_{\phi A} e_m^*\big)\big[MB_\nu\big] \;=\; \sum_{r=0}^{d} \omega f_r^* \big[MB_\nu\big] \; e_{m-r}\big[B_\nu\big] \,. \tag{3.38}$$

Now recall that since $\nu \vdash k$ we shall have $e_{m-r}\big[B_\nu\big] \neq 0$ only when $m-r \leq k$. Since $m = k+d$ this gives $r \geq d$, reducing 3.38 to a single term

$$\big(\Delta_{\phi A} e_m^*\big)\big[MB_\nu\big] = \omega f_d^*\big[MB_\nu\big] \; e_k\big[B_\nu\big] = \omega f_d^*\big[MB_\nu\big] \; T_\nu \,. \tag{3.39}$$

However, since ∇ maps each subspace $\Lambda^{=r}$ onto itself and both translation operators $\mathcal{T}_1, \mathcal{T}_\epsilon$ differ from the identity operator by degree lowering operators (see 3.8 and 3.9), we see from 3.34 that

$$\omega f_d^* \;=\; \nabla A \,.$$

This reduces 3.39 to 3.29 and completes our proof.

4. Proof of the basic recursion.

Let us recall that the developments in sections 1 and 2 have reduced the proof of the recursion in I.26 to a verification of the identity in 2.20 for $1 \le k \le m$. That is

$$\sum_{i=1}^{k} \left[\begin{matrix} k \\ i \end{matrix} \right]_q q^{\binom{i}{2}-i} (1-q^i)\, t^{m-i} \sum_{\mu \vdash m} \frac{T_\mu^2\, \Pi_\mu}{\tilde{h}_\mu \tilde{h}'_\mu}\, e_i[(1-t)\widetilde{B}_\mu] \;=\; = \frac{t^{m-k}}{q^k}\, q^{\binom{k}{2}}\, \nabla e_{m-k}\big[X \tfrac{1-q^k}{1-q}\big] \Big|_{S_{1^{m-k}}} . \qquad 4.1$$

We shall begin by working with the expression

$$E_{m,i}(q,t) \;=\; \sum_{\mu \vdash m} \frac{T_\mu^2\, \Pi_\mu}{\tilde{h}_\mu \tilde{h}'_\mu}\, e_i[(1-t)\widetilde{B}_\mu] \,. \qquad 4.2$$

The idea is to make use of the summation formulas derived in section 3 to decompose 4.2 into a sum of expressions involving partitions of size smaller than m.

We shall thus begin to look for a $g \in \Lambda^{\le i}$ such that

$$T_\mu\, e_i[(1-t)\widetilde{B}_\mu] \;=\; \sum_{\substack{\nu \subseteq \mu \\ m-i \le |\nu| \le m}} c_{\mu,\nu}^{(\omega g)^\perp}(q,t)\, T_\nu \,. \qquad 4.3$$

From Theorem 3.3 we derive that we must then have

$$e_i[(1-t)\widetilde{B}_\mu] \;=\; \omega\nabla\Big(g\big[\tfrac{X+1}{\widetilde{M}}\big]\Big)\big[\widetilde{M}\,\widetilde{B}_\mu - 1\big] \,.$$

This is assured if we solve for g in

$$\omega\nabla\Big(g\big[\tfrac{X+1}{\widetilde{M}}\big]\Big) = e_i\big[(1-t)\tfrac{X+1}{\widetilde{M}}\big] \,.$$

Now, applying ω to both sides we get

$$\begin{aligned} \nabla\Big(g\big[\tfrac{X+1}{\widetilde{M}}\big]\Big) &= e_i\big[(1-t)\tfrac{-\epsilon X+1}{\widetilde{M}}\big] \\ &= (qt)^i\, e_i\big[\tfrac{-\epsilon X+1}{1-q}\big] \\ &= (qt)^i \sum_{r=0}^{i} e_r\big[\tfrac{-\epsilon X}{1-q}\big]\, e_{i-r}\big[\tfrac{1}{1-q}\big] \,, \end{aligned}$$

which may be rewritten as

$$\nabla\Big(g\big[\tfrac{X+1}{\widetilde{M}}\big]\Big) \;=\; (qt)^i \sum_{r=0}^{i} h_r\big[\tfrac{X}{1-q}\big]\, q^{\binom{i-r}{2}} h_{i-r}\big[\tfrac{1}{1-q}\big] \,.$$

We can now apply ∇^{-1} to both sides and obtain

$$g\big[\tfrac{X+1}{\widetilde{M}}\big] \;=\; (qt)^i \sum_{r=0}^{i} q^{-\binom{r}{2}} h_r\big[\tfrac{X}{1-q}\big]\, q^{\binom{i-r}{2}} h_{i-r}\big[\tfrac{1}{1-q}\big]\,,$$

and the relation

$$\binom{i-r}{2}-\binom{r}{2} \;=\; \binom{i}{2}+r(1-i)$$

gives

$$\begin{aligned} g\big[\tfrac{X+1}{\widetilde{M}}\big] &= (qt)^i\, q^{\binom{i}{2}} \sum_{r=0}^{i} h_r\big[\tfrac{q^{1-i}X}{1-q}\big]\, h_{i-r}\big[\tfrac{1}{1-q}\big] \\ &= (qt)^i\, q^{\binom{i}{2}}\; h_i\big[\tfrac{q^{1-i}X+1}{1-q}\big]\,. \end{aligned}$$

Now, setting $X\to\frac{X}{qt}-1$, the following manipulations

$$\begin{aligned} g^* = g\big[\tfrac{X}{M}\big] &= (qt)^i\, q^{\binom{i}{2}}\; h_i\big[\tfrac{q^{1-i}(\frac{X}{qt}-1)+1}{1-q}\big] \\ &= (qt)^i\, q^{\binom{i}{2}}\; h_i\big[\tfrac{\frac{X}{q^i t}+1-\frac{q}{q^i}}{1-q}\big] \\ &= q^{\binom{i}{2}-i^2+i}\; h_i\big[\tfrac{X+t(q^i-q)}{1-q}\big] \end{aligned}$$

yield that

$$g^* = q^{-\binom{i}{2}}\; h_i\big[\tfrac{X+t(q^i-q)}{1-q}\big]\;. \qquad 4.4$$

With this choice of g let us now substitute 4.3 into 4.2 and obtain (using 3.5)

$$\begin{aligned} E_{m,i}(q,t) &= \sum_{\mu\vdash m} \frac{T_\mu \Pi_\mu}{\tilde h_\mu \tilde h'_\mu} \sum_{\substack{\nu\subseteq\mu \\ m-i\le|\nu|\le m}} c_{\mu,\nu}^{(\omega g)^\perp} T_\nu \\ &= \sum_{r=0}^{i} \sum_{\nu\vdash m-r} \frac{T_\nu}{\tilde h_\nu \tilde h'_\nu} \sum_{\substack{\mu\supseteq\nu \\ \mu\vdash m}} d_{\mu,\nu}^{g^*} T_\mu \Pi_\mu\;. \end{aligned} \qquad 4.5$$

But as we observed once before, for $\mu\vdash m$ and $\nu\vdash m-r$ only the homogeneous component of degree r of g^* , which we denote g^*_r, can contribute to the scalar product in the formula

$$d_{\mu,\nu}^{g^*} \;=\; \big\langle \tilde H_\mu\;,\; g^* \tilde H_\mu \big\rangle_* / \tilde h_\mu \tilde h'_\mu\;.$$

We may thus rewrite 4.5 in the form

$$E_{m,i}(q,t) \;=\; \sum_{r=0}^{i} \sum_{\nu\vdash m-r} \frac{T_\nu}{\tilde h_\nu \tilde h'_\nu} \sum_{\substack{\mu\supseteq\nu \\ \mu\vdash m}} d_{\mu,\nu}^{g^*_r} T_\mu \Pi_\mu\;.$$

We can now apply Theorem I.3 to get

$$E_{m,i}(q,t) \;=\; \sum_{r=0}^{i} \sum_{\nu\vdash m-r} \frac{T_\nu^2\,\Pi_\nu}{\tilde h_\nu \tilde h'_\nu}\;(\nabla g_r^*)[MB_\nu]\,. \qquad 4.6$$

To compute ∇g_r^* we go back to 4.4 and obtain

$$g_r^* = q^{-\binom{i}{2}}\; h_r\big[\tfrac{X}{1-q}\big] h_{i-r}\big[\tfrac{t(q^i-q)}{1-q}\big]\;,$$

which gives

$$\begin{aligned}\nabla g_r^* \;&=\; q^{-\binom{i}{2}}\; q^{\binom{r}{2}} h_r\big[\tfrac{X}{1-q}\big] h_{i-r}\big[\tfrac{t(q^i-q)}{1-q}\big]\\ &=\; q^{-\binom{i}{2}}\; q^{\binom{r}{2}} (-tq)^{i-r} h_r\big[\tfrac{X}{1-q}\big] e_{i-r}\big[\tfrac{1-q^{i-1}}{1-q}\big]\\ &=\; q^{-\binom{i}{2}}\; q^{\binom{r}{2}} q^{\binom{i-r}{2}} (-tq)^{i-r} h_r\big[\tfrac{X}{1-q}\big] \big[\begin{smallmatrix} i-1\\ i-r\end{smallmatrix}\big]_q\,.\end{aligned}$$

Thus we finally have

$$\big(\nabla g_r^*\big)[MB_\nu] \;=\; q^{-\binom{i}{2}}\; q^{\binom{r}{2}} q^{\binom{i-r}{2}} (-tq)^{i-r} h_r\big[(1-t)B_\nu\big] \big[\begin{smallmatrix} i-1\\ i-r\end{smallmatrix}\big]_q\,. \qquad 4.7$$

Substituting 4.7 in 4.6 we get

$$\begin{aligned} E_{m,i}(q,t) \;&=\; q^{-\binom{i}{2}} \sum_{r=1}^{i} (-tq)^{i-r} q^{\binom{r}{2}} q^{\binom{i-r}{2}}\; \big[\begin{smallmatrix} i-1\\ i-r\end{smallmatrix}\big]_q \sum_{\nu\vdash m-r} \frac{T_\nu^2\,\Pi_\nu}{\tilde h_\nu \tilde h'_\nu}\; h_r\big[(1-t)B_\nu\big]\\ (\text{ by 2.18 } \rightarrow) \;&=\; q^{-\binom{i}{2}} \sum_{r=1}^{i} (-tq)^{i-r} q^{\binom{r}{2}} q^{\binom{i-r}{2}}\; \big[\begin{smallmatrix} i-1\\ i-r\end{smallmatrix}\big]_q \tfrac{1}{1-q^r}\, \nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big|_{S_{1^{m-r}}}\,.\end{aligned}$$

This given the left hand side of 4.1 becomes

$$\begin{aligned} LHS &= \sum_{i=1}^{k} \Big[\begin{smallmatrix} k\\ i\end{smallmatrix}\Big]_q q^{\binom{i}{2}-i}\,(1-q^i)\,t^{m-i} q^{-\binom{i}{2}} \sum_{r=1}^{i} (-tq)^{i-r} q^{\binom{r}{2}} q^{\binom{i-r}{2}} \big[\begin{smallmatrix} i-1\\ i-r\end{smallmatrix}\big]_q \tfrac{1}{1-q^r} \nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big|_{S_{1^{m-r}}}\\ &= \sum_{r=1}^{k} q^{\binom{r}{2}-r} \tfrac{t^{m-r}}{1-q^r}\, \nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big|_{S_{1^{m-r}}} \sum_{i=r}^{k} \Big[\begin{smallmatrix} k\\ i\end{smallmatrix}\Big]_q (1-q^i)\, q^{\binom{i-r}{2}}\; \big[\begin{smallmatrix} i-1\\ i-r\end{smallmatrix}\big]_q (-1)^{i-r}\,. \qquad 4.8\\ &= \sum_{r=1}^{k} q^{\binom{r}{2}-r} \tfrac{t^{m-r}}{1-q^r}\, \nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big|_{S_{1^{m-r}}} \sum_{u=0}^{k-r} \Big[\begin{smallmatrix} k\\ r+u\end{smallmatrix}\Big]_q (1-q^{r+u})\, q^{\binom{u}{2}}\; \big[\begin{smallmatrix} r+u-1\\ u\end{smallmatrix}\big]_q (-1)^{u}\,.\end{aligned}$$

Denoting by $\mathcal{S}(k,r)$ the sum over u, we see that

$$\begin{aligned}
\mathcal{S}(k,r) &= \sum_{u=0}^{k-r} \frac{[k]_q!(1-q^{r+u})}{[k-r-u]_q![r+u]_q!}\; q^{\binom{u}{2}}\, \frac{[r+u-1]_q!}{[u]_q![r-1]_q!}(-1)^u \\
&= (1-q)\sum_{u=0}^{k-r} \frac{[k]_q!}{[k-r-u]_q!}\; q^{\binom{u}{2}}\, \frac{1}{[u]_q![r-1]_q!}(-1)^u \\
&= (1-q^r)\left[\begin{smallmatrix} k \\ r \end{smallmatrix}\right]_q \sum_{u=0}^{k-r} \frac{[k-r]_q!}{[k-r-u]_q![u]_q!}\; q^{\binom{u}{2}}(-1)^u \\
&= (1-q^r)\left[\begin{smallmatrix} k \\ r \end{smallmatrix}\right]_q \sum_{u=0}^{k-r} \left[\begin{smallmatrix} k-r \\ u \end{smallmatrix}\right]_q\; q^{\binom{u}{2}}(-1)^u \\
&= (1-q^r)\left[\begin{smallmatrix} k \\ r \end{smallmatrix}\right]_q \prod_{u=0}^{k-r-1}(1-q^u) \;=\; \begin{cases} 1-q^k & \text{if } r=k \\ 0 & \text{otherwise}\;. \end{cases}
\end{aligned}$$

Substituting this in 4.8 reduces it to

$$LHS \;=\; q^{\binom{k}{2}-k}\,\frac{t^{m-r}}{1-q^k}\,\nabla e_{m-k}\big[X\tfrac{1-q^k}{1-q}\big]\Big|_{S_{1^{m-k}}}(1-q^k)\;,$$

which equals the right hand side of 4.1. This proves 4.1 and completes our proof of the recursion in I.26.

5. Towards a sectionalization of Diagonal Harmonics.

Let us recall that the space $\mathcal{DH}_n$ of “Diagonal Harmonics” consists of the solutions to the system of differential equations

$$\sum_{i=1}^{n} \partial_{x_i}^h \partial_{y_i}^k P(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_n) \;=\; 0 \qquad 1\le h+k\le n\;. \tag{5.1}$$

It is not difficult to show (see [11]) that the elements of $\mathcal{DH}_n$ are polynomials of total degree bounded by $\binom{n}{2}$. Moreover, since the equations in 5.1 are bihomogeneous, the space $\mathcal{DH}_n$ has a natural bigrading which decomposes it into the direct sum

$$\mathcal{DH}_n \;=\; \bigoplus_{0\le h+k\le\binom{n}{2}} \mathcal{H}_{h,k}(\mathcal{DH}_n)\,,$$

where $\mathcal{H}_{h,k}(\mathcal{DH}_n)$ consists of the elements of $\mathcal{DH}_n$ which are bihomogeneous of degree h in $x_1,x_2,\ldots,x_n$ and degree k in $y_1,y_2,\ldots,y_n$. Note that since the differential equations in 5.1 are invariant under the “*diagonal action*” of S_n (†) we see that each subspace $\mathcal{H}_{h,k}(\mathcal{DH}_n)$ is itself

(†) This is the action defined by setting for $\sigma\in S_n : \sigma P = P(x_{\sigma_1},\ldots,x_{\sigma_n};y_{\sigma_1},\ldots,y_{\sigma_n})$.

invariant under this action. Denoting by $char(\mathcal{H}_{h,k}(\mathcal{DH}_n))$ the corresponding character and by $\mathbf{F}char(\mathcal{H}_{h,k}(\mathcal{DH}_n))$ the Frobenius image of this character, then the symmetric function

$$DH_n[X;q,t] \;=\; \sum_{0\le h+k\le\binom{n}{2}} t^h q^q \mathbf{F}char(\mathcal{H}_{h,k}(\mathcal{DH}_n))$$

may be referred to as the "*bivariate Frobenius characteristic*" of $\mathcal{DH}_n$. In particular the polynomial

$$F_n(q,t) \;=\; \partial_{p_1}^n DH_n[X;q,t]$$

gives the bivariate Hilbert series of $\mathcal{DH}_n$. Likewise, the polynomial

$$DH_n[X;q,t]\,\big|_{S_{1^n}}$$

gives the bivariate Hilbert series of the subspace $\mathcal{DHA}_n$ of Diagonal Harmonic Alternants.

Computer explorations carried out by Haiman in 1990 have led a number of people (see [11]) to conjecture a variety of combinatorial properties of the diagonal action of S_n on the subspaces $\mathcal{H}_{h,k}(\mathcal{DH}_n)$. Hovever, it was later shown [6] that all of these properties may be derived from the single conjecture

$$DH_n[X;q,t] \;=\; \nabla e_n[X]\,. \tag{5.2}$$

The present developments reveal that this identity implies some additional combinatorial properties of the spaces of Diagonal Harmonics which had not been observed nor predicted in previous work. To be more explicit, we infer from the recursion in I.26 that $\mathcal{DH}_n$, as an S_n-module, may admit a further sectionalization into n distinct, S_n invariant "*classes*" (or submodules). In this section we prove some identities which lend further support to this possibility.

To see how this comes about our point of departure is I.26 with $m=n$ and $s=1$. Namely

$$\nabla e_n[X]\,\big|_{S_{1^n}} \;=\; \sum_{r=1}^{n} t^{n-r} q^{\binom{r}{2}} \nabla e_{n-r}[X\tfrac{1-q^r}{1-q}]\,\big|_{S_{1^{n-r}}}\,. \tag{5.3}$$

Setting, as we did in the introduction

$$Q_{n,s}(q,t) \;=\; t^{n-s} q^{\binom{s}{2}} \nabla e_{n-s}[X\tfrac{1-q^s}{1-q}]\big|_{S_{1^{n-s}}}\,, \tag{5.4}$$

5.3 becomes

$$\nabla e_n[X]\,\big|_{S_{1^n}} \;=\; \sum_{r=1}^{n} Q_{n,r}(q,t)\,. \tag{5.5}$$

On the validity of the conjecture in 5.2, this identity suggests that the space $\mathcal{DHA}_n$ of Diagonal Harmonic Alternants may be broken up into n bihomogeneous "*classes*"

$$\mathcal{DHA}_{n,1}\ ,\ \mathcal{DHA}_{n,2}\ ,\ \ldots\ ,\ \mathcal{DHA}_{n,n}$$

with bivariate Hilbert series respectively given by the polynomials

$$Q_{n,1}(q,t)\ ,\ Q_{n,2}(q,t)\ ,\ \ldots\ ,\ Q_{n,n}(q,t)\,.$$

In particular, since 5.4 for $s=1$ becomes

$$Q_{n,1} \quad = \quad t^{n-1}\nabla e_{n-1}[X]\big|_{S_{1^{n-1}}}\ , \tag{5.6}$$

we infer from 5.2 that the "*class*" $\mathcal{DHA}_{n,1}$ should be none other than a bigraded "replica" of $\mathcal{DHA}_{n-1}$, shifted by $n-1$ in the x-degree. This class is easy to identify if we make use of the so-called "*Operator Conjecture*" formulated in [11]. Under this conjecture, $\mathcal{DHA}_n$ is the linear span of the polynomials obtained by successive applications of the operators

$$D_{k,n} \quad = \quad \sum_{i=1}^{n} y_i \partial_{x_i}^{k} \qquad (\text{for } k=1,\cdots,n-1) \tag{5.7}$$

on the Vandermonde determinant

$$\Delta_n(x) \quad = \quad \prod_{1\le i<j\le n}(x_i-x_j)\ .$$

On the validity of this conjecture, let us assume that $\mathcal{B}_{n-1}$ is a set of exponent sequences $p=(p_1,p_2,\ldots,p_{n-2})$ selected so that the collection

$$\Big\{D_{1,n-1}^{p_1}D_{2,n-1}^{p_2}\cdots D_{n-2,n-1}^{p_{n-2}}\Delta_{n-1}(x)\Big\}_{p\in\mathcal{B}_{n-1}} \tag{5.8}$$

is a basis for $\mathcal{DHA}_{n-1}$. This given, it is easy to see that the subspace of $\mathcal{DHA}_n$ spanned by the polynomials

$$\Big\{D_{1,n}^{p_1}D_{2,n}^{p_2}\cdots D_{n-2,n}^{p_{n-2}}\Delta_{n}(x)\Big\}_{p\in\mathcal{B}_{n-1}} \tag{5.9}$$

is an acceptable candidate for the "class" $\mathcal{DHA}_{n,1}$. In fact, its bigraded Hilbert series is necessarily given by 5.6 since the obvious relation

$$\Delta_{n-1}=\tfrac{1}{(n-1)!}\,\partial_{x_n}^{n-1}\Delta_n$$

shows that the operator $\partial_{x_n}^{n-1}$ maps the collection in 5.9 onto the bihomogeneous basis in 5.8. On the other extreme, the classes $\mathcal{DHA}_{n,n-1}$ and $\mathcal{DHA}_{n,n}$ are also easy to identify. Indeed, 5.4 for $s=n-1$ and $s=n$ respectively gives

$$Q_{n,n-1}=t\,q^{\binom{n-1}{2}}\big(1+q+q^2+\cdots+q^{n-2}\big) \quad \text{and} \qquad Q_{n,n}=q^{\binom{n}{2}}\,.$$

This suggests that $\mathcal{DHA}_{n,n}$ is the one dimensional space spanned by the Vandermonde determinant $\Delta_n(y)$ and $\mathcal{DHA}_{n,n-1}$ could be the linear span of the collection

$$\Big\{R_{1,n}\Delta_n(y)\ ,\ R_{2,n}\Delta_n(y)\ ,\ \ldots\ ,\ R_{n-1,n}\Delta_n(y)\Big\}$$

where we have set

$$R_{k,n} = \sum_{i=1}^{n} x_i\, \partial_{y_i}^k \qquad \big(\text{for } k = 1, \ldots, n-1\big)\,.$$

Remark 5.1

We should mention that the space of Diagonal Harmonics is preserved by the family of operators

$$\Big\{ \sum_{i=1}^{n} x_i\, \partial_{x_i}^h \partial_{y_i}^k\,, \quad \sum_{i=1}^{n} y_i\, \partial_{x_i}^h \partial_{y_i}^k \quad : \quad h+k \geq 1 \Big\}\,.$$

Using this fact it is not difficult to show that successive applications of the operators $D_{k,n}$ on $\Delta_n(x)$ or successive applications of the operators $R_{k,n}$ on $\Delta_n(y)$, after taking linear spans, yield the same subspace of Diagonal Harmonic Alternants. Of course, on the operator conjecture this subspace fills $\mathcal{DHA}_n$.

At this point it is compelling to ask whether the same type of sectionalization might not take place for the whole space of Diagonal Harmonics. The nature of this sectionalization should again be dictated by a corresponding decomposition of the polynomial $\nabla e_n[X]$. With this in mind, we were led to search for an identity which would reduce to I.26 upon equating coefficients of S_{1^m}. Now it develops that we may rewrite I.26 in the form

$$\nabla e_m\big[X\tfrac{1-q^s}{1-q}\big]\Big|_{S_{1^m}} = \sum_{r=1}^{m} e_r\big[\tfrac{1-q^s}{1-q}X\big]\Big|_{S_{1^r}} t^{m-r} q^{\binom{r}{2}} \nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big|_{S_{1^{m-r}}}\,. \qquad 5.10$$

Indeed the equality

$$e_r\big[\tfrac{1-q^s}{1-q}X\big]\Big|_{S_{1^r}} = h_r\big[\tfrac{1-q^s}{1-q}\big] = \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q$$

is an immediate consequence of the dual Cauchy formula

$$e_r\big[\tfrac{1-q^s}{1-q}X\big] = \sum_{\lambda\vdash r} S_\lambda\big[\tfrac{1-q^s}{1-q}\big] S_{\lambda'}[X]\,.$$

Since the product of two Schur functions $S_\alpha \times S_\beta$ has no component of the form S_{1^m} unless $\alpha = 1^h$ and $\beta = 1^k$ for some $h+k=m$ we see that 5.10 is equivalent to

$$\Big(\nabla e_m\big[X\tfrac{1-q^s}{1-q}\big] \;-\; \sum_{r=1}^{m} e_r\big[\tfrac{1-q^s}{1-q}X\big] t^{m-r} q^{\binom{r}{2}} \nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big)\Big|_{S_{1^m}} = 0\,. \qquad 5.11$$

If this equality remained true upon removing the symbol "$\Big|_{S_{1^m}}$" we would have the natural extension of I.26 we are seeking. However, the resulting equality can be easily checked by computer to be generally false. Nevertheless, 5.11 brings us to focus on the symmetric function

$$\Gamma_{m,s}[X;q,t] = \nabla e_m\big[X\tfrac{1-q^s}{1-q}\big] - \sum_{r=1}^{m} e_r\big[\tfrac{1-q^s}{1-q}X\big] t^{m-r} q^{\binom{r}{2}} \nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\;. \qquad 5.12$$

To begin with, extensive computer data led us to the following

Conjecture 5.1

For any pair of integers $m, s \geq 1$ *we have an expansion of the form*

$$\Gamma_{m,s}[X;q,t] = (1-t) \sum_{\substack{\lambda \vdash m \\ \lambda \neq 1^m}} K_\lambda^{(m,s)}(q,t)\, S_\lambda[X]$$

with

$$K_\lambda^{(m,s)}(q,t) \in N[q,t]\,.$$

Given the context from which this emerged it is natural to suspect that this Schur positivity is not an accident, but rather reflects some representation theoretical properties of $\mathcal{DH}_n$ as an S_n module. We must leave the quest for such an explanation and/or a proof of the conjecture to further investigations. Nevertheless, it develops that two of its consequences are within reach of the present methods.

Theorem 5.1

For all integers $m, s \geq 1$ *we have*

$$\begin{aligned} &a) \quad \Gamma_{m,s}[X;q,t]\Big|_{S_{1^m}} = 0\,, \\ &b) \quad \Gamma_{m,s}[X;q,1] = 0\,. \end{aligned} \qquad 5.13$$

Proof

Of course, in view of 5.11, 5.13 a) is simply another way of stating the main result of this paper, that is the recurrence in I.26. As for 5.13 b) we can obtain it with surprisingly less effort by means of a remarkable sequence of plethystic manipulations. The point of departure is the trivial identity

$$\Omega\big[-\epsilon X \tfrac{1}{1-q} z\big] \times \Omega\big[\epsilon X \tfrac{q^s}{1-q} z\big] \;=\; \widetilde{\Omega}\big[X \tfrac{1-q^s}{1-q} z\big] \qquad 5.14$$

which follows from the simple equality

$$\Omega[-\epsilon Y] \;=\; \widetilde{\Omega}[Y]$$

after the substitution $Y \to X \frac{1-q^s}{1-q} z$. This given, multiplying both sides of 5.14 by $\Omega\big[\epsilon X \frac{1}{1-q} z\big]$ yields

$$\Omega\big[\epsilon X \tfrac{q^s}{1-q} z\big] \;=\; \widetilde{\Omega}\big[X \tfrac{1-q^s}{1-q} z\big]\; \Omega\big[\epsilon X \tfrac{1}{1-q} z\big]. \qquad 5.15$$

Now it develops, and it is easy to show (see [4] or [3]) that the linear operator ρ which acts on formal power series in z by the linear extension of the map $\rho z^n = q^{\binom{n}{2}}$ transforms a product

$$A(z)\, B(z) \;=\; \sum_{r \geq 0} A_r z^r B(z)$$

into the "tangled" version

$$\rho\big(A(z)B(z)\big) \;=\; \sum_{r\geq 0} A_r z^r q^{\binom{r}{2}} \rho B(zq^r)\,.$$

Applying ρ to both sides of 5.15 then yields

$$\rho\Omega\big[\epsilon X \tfrac{q^s}{1-q} z\big] \;=\; \sum_{r\geq 0} e_r\big[X\tfrac{1-q^s}{1-q}\big] z^r q^{\binom{r}{2}}\, \rho\Omega\big[\epsilon X \tfrac{q^r}{1-q} z\big]\,. \tag{5.16}$$

Our next step is to rewrite this in terms of the operator ${}^q\nabla$ defined by setting for the schur function basis

$${}^q\nabla S_\lambda \;=\; \nabla S_\lambda\,\big|_{t=1}\,.$$

To this end we recall from [3] that we have for any two symmetric functions P, Q

$$\begin{aligned} &a) \qquad {}^q\nabla(PQ) \;=\; (\,{}^q\nabla P)(\,{}^q\nabla Q)\,,\\ &b) \qquad {}^q\nabla h_r\big[\tfrac{X}{1-q}\big] \;=\; q^{\binom{r}{2}} h_r\big[\tfrac{X}{1-q}\big]\,. \end{aligned} \tag{5.17}$$

This is an immediate consequence of the definition I.8 of ∇ combined with the special evaluation (see [6])

$$\tilde H_\mu[X;q,1\,] \;=\; \prod_i h_{\mu_i}\big[\tfrac{X}{1-q}\big](q;q)_{\mu_i}\,.$$

Now we get

$$\begin{aligned} \rho\Omega\big[\epsilon X\tfrac{q^s}{1-q}z\big] &= \sum_{k\geq 0}(\epsilon z q^s)^k q^{\binom{k}{2}} h_k\big[\tfrac{X}{1-q}\big]\\ &= \sum_{k\geq 0}(\epsilon z q^s)^k\;{}^q\nabla h_k\big[\tfrac{X}{1-q}\big]\\ &= \;{}^q\nabla\Omega\big[\epsilon X\tfrac{q^s}{1-q}z\big]\,. \end{aligned} \tag{5.18}$$

Using 5.18 on both sides of 5.16 we obtain

$${}^q\nabla\Omega\big[\epsilon X\tfrac{q^s}{1-q}z\big] \;=\; \sum_{r\geq 0} e_r\big[X\tfrac{1-q^s}{1-q}\big] z^r q^{\binom{r}{2}}\;\,{}^q\nabla\Omega\big[\epsilon X\tfrac{q^r}{1-q}z\big]\,,$$

or equivalently (since $\Omega[\epsilon Y] = \widetilde\Omega[-Y]$):

$${}^q\nabla\widetilde\Omega\big[X\tfrac{-q^s}{1-q}z\big] \;=\; \sum_{r\geq 0} e_r\big[X\tfrac{1-q^s}{1-q}\big] z^r q^{\binom{r}{2}}\;\,{}^q\nabla\widetilde\Omega\big[X\tfrac{-q^r}{1-q}z\big]\,.$$

Multiplying both sides by ${}^q\nabla\widetilde\Omega\big[\tfrac{X}{1-q}z\big]$ and using the multiplicativity of ${}^q\nabla$ (that is, 5.17 a)) we get

$${}^q\nabla\widetilde\Omega\big[X\tfrac{1-q^s}{1-q}z\big] \;=\; \sum_{r\geq 0} e_r\big[X\tfrac{1-q^s}{1-q}\big] z^r q^{\binom{r}{2}}\;\,{}^q\nabla\widetilde\Omega\big[X\tfrac{1-q^r}{1-q}z\big]\,,$$

and equating coefficients of z^{n-s} we finally obtain

$$ {}^{q}\nabla e_{n-s}\big[X\tfrac{1-q^s}{1-q}\big] \;=\; \sum_{r=1}^{n-s} e_r\big[X\tfrac{1-q^s}{1-q}\big]q^{\binom{r}{2}}\ {}^{q}\nabla e_{n-s-r}\big[X\tfrac{1-q^r}{1-q}\big]\,. \qquad 5.19 $$

This proves 5.13 b) since the difference of the two sides of 5.19 is precisely what the right hand side of 5.12 reduces to when we set $t=1$.

An immediate corollary of this identity is a combinatorial interpretation of the symmetric functions

$$ \Phi_{n,s}[X;q] \;=\; q^{\binom{s}{2}}\ {}^{q}\nabla e_{n-s}\big[X\tfrac{1-q^s}{1-q}\big]\,. \qquad 5.20 $$

Theorem 5.2

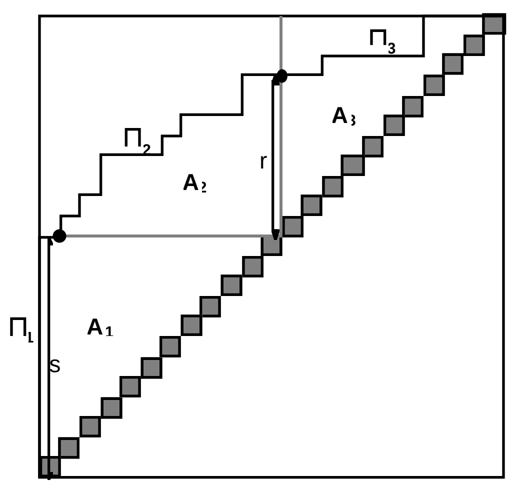
For all pairs of integers $s \leq n$ we have

$$ \Phi_{n,s}[X;q\,] = \sum_{\Pi\in\mathcal{D}_{n,s}} q^{area(\Pi)} e_{m_1(\Pi)}e_{m_2(\Pi)}\cdots e_{m_{n-1}(\Pi)}\ , \qquad 5.21 $$

where the subscript $m_i(\Pi)$ denotes the length of the vertical segment of Π on the vertical line of abscissa i.

Proof

For a moment, let us denote the right hand side of 5.18 by $\Psi_{n,s}[X;q]$. We aim to show that $\Psi_{n,s}[X;q]$ and $\Phi_{n,s}[X;q\,]$ satisfy the same recurrence with the same initial conditions. To this end note that a path $\Pi\in\mathcal{D}_{n,s}$ may be decomposed in three pieces Π_1,Π_2,Π_3 (see figure below).



Here Π_1 consists of the vertical segment of length s and abscissa 0 followed by the first EAST step, Π_2 is the portion of Π that goes from $(1,s)$ to $(s,s+r)$ (for some $1\leq r\leq n-s$) and finally Π_3 is the portion joining $(s,s+r)$ to (n,n). Likewise, the contribution to the area statistic of Π may be decomposed into three parts A_1,A_2,A_3, where $A_1=\binom{s}{2}$ gives the area below the horizontal line $y=s$, A_2 gives the area above $y=s$ and to the left of the vertical line $x=s$

and finally A_3 gives the area to the right of $x=s$. Note that if $m_1, m_2, \ldots, m_s$ are the lengths of the vertical segments of Π on the vertical lines of abscissas $1, 2, \ldots, s$ then we will have

$$m_1+m_2+\cdots+m_s=r \qquad \text{and} \qquad A_2=m_{s-1}+2m_{s-2}+\cdots+(s-1)m_1\,.$$

Finally, if we adjoin to Π_3 the vertical segment from (s,s) to $(s,s+r)$ then, by a slight abuse of notation, we may view the resulting path Π_3' as an element of $\mathcal{D}_{n-s,r}$, and A_3 then gives the area under Π_3'. Putting all this together we see that the symmetric functions $\Psi_{n,s}[X;q]$ statisfy the recursion

$$\begin{aligned}\Psi_{n,s}[X;q] &= q^{\binom{s}{2}}\sum_{r=1}^{n-s}\sum_{m_1+m_2+\cdots+m_s=r} q^{\sum_{i=1}^{s}(s-i)m_i} e_{m_1}[X]\,e_{m_2}[X]\cdots e_{m_s}[X]\,\Psi_{n-s,r}[X;q]\\ &= q^{\binom{s}{2}}\sum_{r=1}^{n-s}\sum_{m_1+m_2+\cdots+m_s=r} e_{m_1}[q^{s-1}X]\,e_{m_2}[q^{s-2}X]\cdots e_{m_s}[X]\,\Psi_{n-s,r}[X;q]\\ &= q^{\binom{s}{2}}\sum_{r=1}^{n-s} e_r[X\tfrac{1-q^s}{1-q}]\,\Psi_{n-s,r}[X;q]\,.\end{aligned}$$

On the other hand, multiplying 5.19 by $q^{\binom{s}{2}}$, from 5.20 we get

$$\Phi_{n,s}[X;q] = q^{\binom{s}{2}}\sum_{r=1}^{n-s} e_r[X\tfrac{1-q^s}{1-q}]\,\Phi_{n-s,r}[X;q]\,,$$

which is precisely the same recurrence. As for the initial conditions, the right hand side of 5.20 for $s=n$ reduces to $q^{\binom{n}{2}}$ and so does the right hand side of 5.21, thus

$$\Phi_{n,n}[X;q]=\Psi_{n,n}[X;q] \qquad \forall\ n\geq 1\,.$$

This completes the proof of the theorem.

We should mention that 5.21 extends a result first established in [6] which gives a similar combinatorial interpretation for the symmetric function ${}^q\nabla e_n$. In fact, as a corollary of Theorem 5.2 we derive that

Theorem 5.3

For all $n\geq 1$

$$ {}^q\nabla e_n[X] \;=\; \sum_{\Pi\in\mathcal{D}_n} q^{area(\Pi)} e_{m_0(\Pi)} e_{m_1(\Pi)}\cdots e_{m_{n-1}(\Pi)}\,. \tag{5.22}$$

In particular

$$ {}^q\nabla e_n[X] \;=\; \sum_{r=1}^{n} e_r\big[X\big]\Phi_{n,r}[X;q]\,. \tag{5.23}$$

Proof

Note that since $m_o(\Pi)$ can take any value between 1 and n, by grouping terms according to this value the right hand side of 5.22 may be rewritten as

$$\begin{aligned} RHS &= \sum_{r=1}^{n} e_r \sum_{\Pi\in\mathcal{D}_{n,r}} q^{area(\Pi)} e_{m_1(\Pi)} e_{m_2(\Pi)} \cdots e_{m_{n-1}(\Pi)} \\ (\text{by } 5.21)\rightarrow \quad &= \sum_{r=1}^{n} e_r[X]\Phi_{n,r}[X;q]\,. \end{aligned} \qquad 5.24$$

On the other hand using the decomposition 5.23 in 5.19 with $n\rightarrow n+1$ and $s\rightarrow 1$ gives

$$\begin{aligned} {}^q\nabla e_n[X] &= \sum_{r=1}^{n} e_r[X] q^{\binom{r}{2}}\ {}^q\nabla e_{n-r}[X\tfrac{1-q^r}{1-q}] \\ (\text{by } 5.20)\rightarrow \quad &= \sum_{r=1}^{n} e_r[X]\Phi_{n,r}[X;q]\,, \end{aligned} \qquad 5.25$$

which is 5.23. Combining 5.23 and 5.24 proves 5.22.

We terminate by posing two interesting problems that stem from the present developments. To begin with 5.5 and 5.23 suggest that the whole module of Diagonal Harmonics must admit a sectionalization in n classes. As we have seen 5.5 should correspond to a decomposition of the Diagonal Harmonic Alternants and we may interpret 5.23 as reflecting a decomposition of all Diagonal Harmonics, *when we ignore the x-degree*. This given, we believe that there should be some way of decomposing ∇e_n itself without any specializations. Most probably, performing such a decomposition will require some new types of operations to further unravel the action of ∇. Efforts in this direction may not only bring some light on the study of Diagonal Harmonics but also enrich our package of tools to deal with symmetric function identities.

The other problem which emerges from our work is also gravid with combinatorial implications. In fact we see that the definition of the q,t-Catalan immediately gives that $C_n(q,t) = C_n(t,q)$. Thus our proof of I.22 implies that we must have

$$\sum_{\Pi\in\mathcal{D}_n} q^{area(\Pi)} t^{maj(\beta(\Pi))} \quad = \quad \sum_{\Pi\in\mathcal{D}_n} t^{area(\Pi)} q^{maj(\beta(\Pi))}\,.$$

A purely combinatorial proof of this identity should lead to interesting findings.

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