Three Faces of the Delta Conjecture

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Let $X_n = \{x_1, \ldots, x_n\}$, $Y_n = \{y_1, \ldots, y_n\}$ be sets of variables. Let

$$
DR_n = \mathbb{C}[X_n, Y_n]/\left\{ \sum_i x_i^a y_i^b : a, b \geq 0, a + b > 0 \right\}
$$

be the ring of diagonal coinvariants. $S_n$ acts “diagonally” on $DR_n$ by permuting the $X$ and $Y$ variables in the same way.

**Example: $n = 2$**

Cosets $\{1, x_1, y_1\}$ form a basis for $DR_2$, so $\text{Hilb}(DR_2) = 1 + q + t$.

The identity in $S_2$ acts by fixing all the cosets, while $\sigma = (12)$ fixes $1$ and sends $\{x_1, y_1\}$ to $\{x_2, y_2\}$. Since $x_1 + x_2 = 0 = y_1 + y_2$, $x_2 = -x_1, y_2 = -y_1$. Hence the coset $1$ corresponds to the trivial character, while $x_1, y_1$ correspond to the sign character, and the bigraded character of $DH_2$ is $s_2 + (q + t)s_{1,1}$. 
The Symmetric Function Side

Let $\Delta'_f$ be a linear operator defined via

$$\Delta'_f \tilde{H}_\mu(X; q, t) = f[B_\mu - 1] \tilde{H}_\mu(X; q, t),$$

where $B_\mu = \sum_{s \in \mu} q^{\text{coarm}(s)} t^{\text{coleg}(s)}$. For example $B_{3,2} = 1 + q + q^2 + t + tq$.

Haiman proved that the bigraded character of $\text{DR}_n$ under the diagonal action is given by

$$\Delta'_{e_{n-1}e_n}(X) = \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu(X; q, t) MB_\mu \prod_{s \in \mu}'(1 - q^{\text{coarm}(s)})(1 - t^{\text{coleg}(s)})}{\prod_{s \in \mu}(t^{\text{leg}(s)} - q^{\text{arm}(s)+1})(q^{\text{arm}(s)} - t^{\text{leg}(s)+1})}.$$ 

where $M = (1 - q)(1 - t)$ and $T_\mu = t^{n(\mu)} q^{n(\mu')}$, with $n(\mu) = \sum_i (i - 1) \mu_i$. 

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Given a Dyck path $\pi$ and a word parking function $P$ (a filling of the squares just to the right of North steps of $\pi$ with cars, i.e. integers between 1 and $n$, strictly increasing up columns), let $a_i$ be the number of area squares in the $i$th row (from the bottom). Cars in rows $(i, j)$ with $i < j$ form an inversion pair if either $a_i = a_j$ and $\text{car}_i < \text{car}_j$, or $a_i = a_j + 1$ and $\text{car}_i > \text{car}_j$. Let $d_i$ be the number of inversion pairs $(i, j)$ with $i < j$. Furthermore, we call a car at the bottom of a column a valley, and say the valley is moveable if, when we slide the car one square to the left, the result is still a word parking function, i.e. we still have strict decrease down columns. For example, in Figure 1, cars 1, 2 and 8 (in rows 5, 6 and 8) are moveable, but cars 4 and 3 in rows 1 and 2 are not.
Figure: A word parking function with area $= 6$. There are dinv $(i,j)$-row pairs $(7, 8), (5, 7), (5, 8), (4, 5), (4, 7), (3, 6), (3, 8)$, so dinv $= 7$. The total weight is $x_1 x_2 x_3 x_4^2 x_7 x_8^2 q^7 t^6$. 
Figure: The various word parking functions when $n = 2$, together with their $x, q, t$ weights.
Theorem (Carlsson-Mellit, 2015)

\[ \Delta_{e_{n-1}}' e_n = \sum_{P \in WP(n)} q^{d \text{inv}(P)} t^{\text{area}(P)} \chi^P. \]

where the sum is over all word parking functions \( P \) on \( n \) cars.

Still Open: Find a combinatorial expression for the Schur expansion of the right-hand-side above.

Corollary (Conjectured by H., Loehr in 2002)

\[ \text{Hilb}(DR_n) = \sum_{\sigma \in S_n} t^{\text{maj}(\sigma)} \prod_{i=1}^{n-1} [w_i(\sigma)]_q. \]
Let $w_i(\sigma)$ equal the number of $w_j$ which are in $\sigma_i$’s run and larger than $\sigma_i$, or in the next run to the right and smaller than $\sigma_i$.

**Example**

$$\sigma = 25713846 \rightarrow 257|138|46|0$$

$$(w_1, w_2, \ldots, w_8) = (3, 3, 2, 2, 1, 2, 2, 1).$$
Theorem (Carlsson-Oblomkov, 2018)

A monomial basis for $\text{DR}_n$ is given by a certain family of cosets, one for each $\sigma \in S_n$. The contribution to $\text{Hilb}(\text{DR}_n)$ of monomials associated to $\sigma$ is $t^{\text{maj}(\sigma)} \prod_{i=1}^{n-1} [w_i(\sigma)]_q$.

Examples

$$\sigma = 25713846 \rightarrow y_1y_2y_3 \times y_1y_2y_3y_4y_5y_6$$

$$(1 + x_2 + x_2^2)(1 + x_5 + x_5^2)(1 + x_7)(1 + x_1)(1 + x_8)(1 + x_4)$$

Set all $x_i = 0$; $\sum_{\sigma \in S_n} \prod_{k \in \text{Des}} y_1y_2 \cdots y_k \rightarrow \text{Garsia-Stanton basis}$

Set all $y_i = 0$; $\sigma = (12 \cdots n) : (w_1, w_2, \ldots) = (n, n - 1, \ldots) \rightarrow (1 + x_1 + \ldots x_1^{n-1}) \cdots (1 + x_{n-2} + x_{n-2}^2)(1 + x_{n-1}) \rightarrow \text{Artin basis.}$
The Delta Conjecture (H., Remmel, Wilson, 2015)

\[ \Delta'_{e_{k-1}} e_n = \sum_{P \in WP(n)} q^{dinv(P)} t^{area(P)} \prod_{a_i > a_{i-1}} \left( 1 + z/t^a_i \right) \bigg|_{z^{n-k}} \]

\[ = \sum_{P \in WP(n)} q^{dinv(P)} t^{area(P)} \prod_{\text{movable valleys}} \left( 1 + z/q^{d_i+1} \right) \bigg|_{z^{n-k}} \]
Let $\Pi$ be an ordered set partition of $\{1, 2, \ldots, n\}$, and let $\sigma = \sigma(\Pi)$ be the ordering of the blocks of $\Pi$ which minimizes $\text{maj}$. For example, if $\Pi = \left\{\{2, 3, 5\}, \{1, 6, 7, 9\}, \{4, 8\}\right\}$, then $\sigma(\Pi) = 235679148$, and $\text{minimaj}(\Pi) = \text{maj}(\sigma) = 6$. Next form $\sigma^*$ by marking every number which is not leftmost (in minimaj order) from its block;

$$\sigma^* = 23^*5^*67^*9^*1^*48^*.$$ 

Now construct the vector $(w_1(\Pi), w_2(\Pi), \ldots)$ by first isolating the unmarked elements of $\sigma^*$, map them to a permutation, and apply previous rule:

$$264 \rightarrow 132 \rightarrow 13|2|0 \rightarrow (1, 1, 1).$$

For marked elements $\sigma^*_i$, $w_i$ equals the number of unmarked elements smaller than $\sigma_i$ in its run plus the number of unmarked elements which are larger in the previous run.

$$\sigma^* = \{23^*5^*\} \{67^*9^*1^*\} \{48^*\} \rightarrow (1, 1, 1, 1, 2, 2, 2, 1, 1).$$
Theorem H.-Sergel, 2018

\[
\sum_{P \in \text{PF}(n)} q^{d_{\text{inv}}(P)} t^{\text{area}(P)} \prod_{\text{movable valleys}} (1 + \frac{z}{q^{d_i+1}})_{z^n-k} = \\
\sum_{\Pi \text{ k blocks}} t^{\text{minimaj}(\Pi)} \prod_{i=1}^{n} [w_i(\Pi)]_q.
\]

Open Question: Is there an analogue involving the rise version of the Delta Conjecture?
M. Zabrocki has recently introduced a module whose bigraded character is conjecturally equal to the combinatorial and symmetric function sides of the Delta Conjecture. Let $\Theta_n = \{\theta_1, \ldots, \theta_n\}$ be a set of anticommuting variables, i.e. $\theta_i \theta_j = -\theta_j \theta_i$, $1 \leq i \leq j \leq n$. Note this implies $\theta_i^2 = 0$. Let $X_n, Y_n$ be two sets of commuting variables, which also commute with the $\theta_i$. Set

$$TR_n = \mathbb{C}[X_n, Y_n, \Theta_n]/\left\{ \sum_{i} x_i^a y_i^b \theta_i^c : a, b, c \geq 0, a + b + c > 0, c \leq 1 \right\}.$$

$S_n$ acts on $TR_n$ diagonally by permuting the $x_i, y_i, \theta_i$ in the same way. Then Zabrocki conjectures that the tri-graded character of this action is given by

$$\sum_{k=1}^{n} z^{n-k} \Delta_{e_{k-1}}' e_n,$$

where $q$, $t$ give the grading in the $x$ and $y$ variables and $z$ the grading in the $\theta$ variables.