The Monotone Column Permanent Conjecture and Multivariate Eulerian Polynomials

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Polynomials with only Real Zeros

Let $f(z) = b_0 + b_1 z + \ldots + b_n z^n$ be a polynomial with real coefficients and only real zeros. Newton proved that

$$b_k^2 \geq b_{k-1} b_{k+1} \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right).$$

As a consequence one has

$$b_k^2 \geq b_{k-1} b_{k+1} \quad \text{(log concavity)}$$

and

$$b_0 \leq b_1 \leq \cdots \leq b_i \geq b_{i+1} \geq \cdots \geq b_n \quad \text{for some } i \text{ (unimodality).}$$

Theorem

Aissen, Schoenberg, and Whitney; Erdei

If $f(z) = b_0 + b_1 z + \cdots + b_n z^n$ *has nonnegative coefficients, then* $f(z)$ *has only real zeros iff all the minors of the infinite matrix below are nonnegative.*

$$\begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_n & 0 & 0 & \cdots \\ 0 & b_0 & b_1 & \cdots & b_{n-1} & b_n & 0 & \cdots \\ 0 & 0 & b_0 & \cdots & b_{n-2} & b_{n-1} & b_n & \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & \end{bmatrix}$$

Example

The polynomials

$$\begin{aligned}
\alpha &= z \\
\beta &= z + z^2 \\
\gamma &= z + 4z^2 + z^3 \\
\delta &= z + 11z^2 + 11z^3 + z^4 \\
\varepsilon &= z + 26z^2 + 66z^3 + 26z^4 + z^5 \\
\zeta &= z + 57z^2 + 302z^3 + 302z^4 + 57z^5 + z^6 \text{ etc."}
\end{aligned}$$

appeared in Euler's work on summation of series. They are now known as *Eulerian polynomials*, and can be expressed as

$$A_n(z) = \sum_{\sigma \in S_n} z^{\mathrm{des}(\sigma)},$$

where des(σ) is the number of descents of σ, i.e. the number of valuers of i for which $\sigma_i > \sigma_{i+1}$.

For example

$$\begin{array}{ccc} 123 \to 1 & 132 \to z & 213 \to z \\ 231 \to z & 312 \to z & 321 \to z^2 \end{array}$$

Letting $A_{n,k}$ denote the number of elements in S_n with k descents, we have

$$A_{n,k} = A_{n-1,k}(k+1) + A_{n-1,k-1}(n-k),$$

which is equivalent to

$$A_n(z) = A_{n-1}(z)(1+(n-1)z) + A'_{n-1}(z)\big(z - z^2\big).$$

This recurrence and the method of interlacing roots can be used to prove $A_n(z)$ has only real zeros.

Theorem

(Brändén, H., Visontai, Wagner (2009); the Monotone Column Permanent (MCP) Theorem) Let C be an $n \times n$ matrix of real numbers, weakly increasing down columns, i.e. $c_{ij} \leq c_{i+1,j}$. Then as a polynomial in z,

$$per(C + zJ_n)$$

has only real zeros. Here per is the permanent, and J_n the matrix of all 1's.

Example

$n = 2$

$$\text{per}\begin{bmatrix} a+z & c+z \\ b+z & d+z \end{bmatrix} = (a+z)(d+z) + (b+z)(c+z).$$

Assume WLOG that $a = 0$; replace d by $c + d$, and so for $b, c, d \geq 0$, we need the discriminant of

$$z(c+d+z) + (b+z)(c+z) = 2z^2 + z(2c+b+d) + bc$$

to be nonnegative. The discriminant is

$$\begin{aligned}(2c+b+d)^2 - 8bc &= 4c^2 + b^2 + d^2 + 4bc + 4cd + 2bc - 8bc \\ &= (2c-b)^2 + b^2 + d^2 + 4cd + 2bc.\end{aligned}$$

The MCP implies the fact that the Eulerian polynomials have only real zeros;

$$A_n(z) = \text{per}\begin{bmatrix} 1 & 1 & \cdots 1 & 1 & 1 \\ 1 & 1 & \cdots 1 & 1 & z \\ 1 & 1 & \cdots 1 & z & z \\ \vdots & \vdots & & \vdots & \\ 1 & z & \cdots z & z & z \end{bmatrix} \sim \text{per}\begin{bmatrix} z & z & \cdots z \\ z & z & \cdots z+1 \\ \vdots & \vdots & \\ z & z & \cdots z+1 \\ z & z+1 & \cdots z+1 \end{bmatrix}$$

$$\sigma = 3152674, \quad \beta(\sigma) = (31)(52)(674).$$

Place rooks on squares $(3,1),(1,3),(5,2),(2,5),(6,7),(7,4),(4,6)$. Rooks below diagonal correspond to descents.

The MCP was conjectured in 1996 by H., Ono, and Wagner. The $n = 3$ case was proved by Ray Mayer shortly after, and two proofs of the $n = 4$ case appeared in 2009. One, due to H. and Visontai, utilizes Brändén and Borcea's results on *stability*. It is based on the following stronger form of the MCP, we call the MMCP.

Conjecture

(H., Visontai 2009). Let $z_1, z_2, \ldots, z_m$ be complex parameters, and C a column monotone $n \times m$ matrix of real numbers. Then for $k \leq min(n, m)$,

$$per_k(c_{ij} + z_j)$$

is stable, i.e. is nonzero if all the z_j are in the upper half-plane. Here per_k is the sum of all $k \times k$ permanental minors.

Theorem

(Brändén; 2007) Let $f(z_1, \ldots, z_n)$ be a linear polynomial with real coefficients. Then f is stable iff

$$f_i(\mathbf{a})f_j(\mathbf{a}) - f(\mathbf{a})f_{ij}(\mathbf{a}) \geq 0, \quad \textit{for all } 1 \leq i < j \leq n \textit{ and all } \mathbf{a} \in \mathbb{R}^n,$$

in which the subscripts denote partial differentiation.

Example

The $n = m = k = 2$ case of the MMCP.

$$\begin{aligned} f &= \text{per} \begin{bmatrix} a+z & c_1+w \\ b+z & d_1+w \end{bmatrix} \\ &= 2zw + z(c+d) + w(a+b) + ad + bc \\ f_z &= 2w + c + d \quad f_w = 2z + a + b \\ f_z f_w - ff_{zw} &= (d-c)(b-a) \geq 0. \end{aligned}$$

Theorem

(Brändén, H., Visontai, Wagner (2009). The MMCP Conjecture is true if $n \geq m$.

Proof (for the case $k = m, n \geq m$). First we show that if there is a counterexample to the MMCP, there is one where each column has at most two different c_{ij} entries. Clearly the MMCP is true if $m = 1$. Assume we have a counterexample to the MMCP, for minimal m, with some column, say the first, containing at least 3 different $c_{i,1}$ entries;

$$\mathrm{per}\begin{bmatrix} a+z_1 & \cdots \\ a+z_1 & \cdots \\ b+z_1 & \cdots \\ c+z_1 & B_{n,m-1} \\ \vdots & \vdots \\ c+z_1 & \cdots \end{bmatrix} = z_1 \mathrm{per}(B_{n,m-1})(n-m+1) + bw + w'.$$

By assumption, $\Im(z_1) > 0$. View z_1 as a function of b, and let b vary from a to c. Since z_1 moves along a line segment in $\mathbb{C}$, it must be in the upper half-plane at one of the ends of this segment. Now replace z_1 by $z_1 - a$, then z_1 by cz_1, resulting in

$$\begin{bmatrix} z_1 & \cdots \\ z_1 & \cdots \\ c + z_1 & \cdots \\ \vdots & \vdots \\ c + z_1 & \cdots \end{bmatrix} \to \begin{bmatrix} cz_1 & \cdots \\ cz_1 & \cdots \\ c + cz_1 & \cdots \\ \vdots & \vdots \\ c + cz_1 & \cdots \end{bmatrix} \to \begin{bmatrix} z_1 & \cdots \\ z_1 & \cdots \\ 1 + z_1 & \cdots \\ \vdots & \vdots \\ 1 + z_1 & \cdots \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & \cdots \\ 1 & \cdots \\ 1 + 1/z_1 & \cdots \\ \vdots & \vdots \\ 1 + 1/z_1 & \cdots \end{bmatrix} \to \begin{bmatrix} 1 & \cdots \\ 1 & \cdots \\ z_1 & \cdots \\ \vdots & \vdots \\ z_1 & \cdots \end{bmatrix}$$

Proposition

For any $n \times n$ Ferrers board F, let

$$F(X,Y) = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & \\ x_1 & x_2 & x_3 & \cdots & \\ \vdots & \vdots & & & \\ x_1 & x_2 & y_{k+1} & y_{k+1} & \cdots \\ y_k & y_k & y_k & y_k & \cdots \\ \vdots & \vdots & & & \\ y_2 & y_2 & y_2 & y_2 & \cdots \\ y_1 & y_1 & y_1 & y_1 & \cdots \end{bmatrix}.$$

Then per$(F(X,Y))$ is stable in $x_1, \ldots, x_n, y_1, \ldots, y_n$.

Corollary

The MMCP is true if $n \geq m$.

Proof of the Proposition: Let $F^* = F^*(X, Y)$ denote $F(X, Y)$ with the first column and bottom row removed. Then

$$\begin{aligned} \mathrm{per}(F(X,Y)) &= x_1\Big(\sum_{i=2}^{n} \partial x_i + \sum_{i=k+1}^{n} \partial y_i\Big)\mathrm{per}(F^*) + ky_1\mathrm{per}(F^*) \\ &= (x_1\partial^* + k)\mathrm{per}(F^*)y_1, \end{aligned}$$

where

$$\partial^* = \sum_{i=2}^{n} \partial x_i + \sum_{i=k+1}^{n} \partial y_i.$$

The operator $x_1\partial^* + k$ is known to preserve stability, so the result follows by induction. (If first column of F is all 0's, flip $F(X, Y)$ upside down and rotate).

Corollary

The Multivariate Eulerian polynomial

$$A_n(x_2,\dots,x_n,y_2,\dots,y_n) = \sum_{\sigma\in S_n} \prod_{\sigma_i>\sigma_{i+1}} x_{\sigma_i} \prod_{\sigma_i<\sigma_{i+1}} y_{\sigma_{i+1}}$$

is stable in $x_2,\dots,x_n,y_2,\dots,y_n$.

Corollary

(Stable version of Simion's result). Let $M(\mathbf{v})$ *denote the set of permutations of the multiset* $\{1^{v_1}2^{v_2}\cdots k^{v_k}\}$. *Then*

$$\sum_{\sigma\in M(\mathbf{v})} \prod_{\sigma_i>\sigma_{i+1}} z_{\sigma_i} \qquad (1)$$

is stable in $z_2,\dots,z_k$.

Corollary

Let $Top(i;n)$ denote the number of permutations in S_n that have i as a "descent top" (where i is immediately followed by something less than i). Then for all $1 \leq i < j \leq n$,

$$\frac{Top(i;n)}{n!}\frac{Top(j;n)}{n!} \geq \frac{Top(i,j;n)}{n!}.$$

This shows that occurrences of descent tops in random permutations are negatively correlated.

Let Q_k denote the set of "Stirling permutations", that is permutations β of $\{0,0,1,1,2,2,\dots,k,k\}$ with the property that all the numbers between any two occurrences of the number j are larger than j, and which begin and end with 0. Gessel and Stanley showed that

$$\sum_{n=0}^{\infty} S(n+k,n) z^n = \Big(\sum_{\beta \in Q_k} z^{\mathrm{des}(\beta)}\Big)/(1-z)^{2k+1}, \qquad (2)$$

where $S(n,k)$ is the Stirling number of the second kind.

Example

$$Q_2 = \{011220, \quad 012210, \quad 022110\}$$
$$A_3^{(2)}(z) = z + 2z^2.$$

Note $|Q_k| = 1 * 3 * 5 * \cdots * (2k-1)$.

For $\beta \in Q_k$, let $\mathrm{Des}(\beta)$ denote the set of *descents* (values of i for which $\beta_i > \beta_{i+1}$) $\mathrm{Asc}(\beta)$ the set of *ascents* (values of i for which $\beta_{i-1} < \beta_i$) and $\mathrm{Plat}(\beta)$ the set of *plateaux* (values of i for which $\beta_i = \beta_{i+1}$).

Theorem

(H., Visontai 2011) The polynomial

$$A_n^{(2)}(X, Y, Z) = \sum_{\beta \in Q_n} \prod_{i \in Des} x_{\beta_i} \prod_{i \in Asc} y_{\beta_i} \prod_{i \in Plat} z_{\beta_i}.$$

is stable, and satisfies

$$A_n^{(2)}(X, Y, Z) = x_n y_n z_n \, \partial \, A_{n-1}^{(2)}(X, Y, Z),$$

where

$$\partial = \sum_{i=1}^{n-1} \partial x_i + \partial y_i + \partial z_i.$$

A version of the previous theorem holds for r-Stirling permutations, for any $r \geq 1$, which were also introduced by Gessel and Stanley. They are permutations of the multiset

$$\{0^2 1^r 2^r \cdots k^r\}$$

which begin and end with 0, and have the property that all numbers between any two occurrences of j are at least j. In particular we have

$$A_n^{(1)}(X,Y) = \sum_{\substack{\sigma \in S_n \\ \sigma_0=0,\ \sigma_{n+1}=0}} \prod_{\sigma_i > \sigma_{i+1}} x_{\sigma_i} \prod_{\sigma_{i-1} < \sigma_i} y_{\sigma_i}$$

is stable in $x_1, \dots, x_n, y_1, \dots, y_n$, and satisfies

$$A_n^{(1)}(X,Y) = x_n y_n \left(\sum_{i=1}^{n-1} \partial x_i + \partial y_i \right) A_{n-1}^{(1)}(X,Y).$$

Let

$$f(z) = \sum_{k=0}^{n} a_k z^k, \qquad g(z) = \sum_{k=0}^{n} b_k z^k.$$

We say f and g are *apolar* if

$$\sum_{k=0}^{n} a_k b_{n-k} \binom{n}{k} (-1)^k = 0.$$

A circular domain in $\mathbb{C}$ is the closed interior or closed exterior of a circle, or a half-plane.

Theorem

(Grace's Apolarity Theorem) If f, g are apolar, then any circular domain which contains all the roots of f contains at least one of the roots of g.

If $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ are the roots of f and g, respectively, then the apolarity condition is equivalent to $\mathrm{per}(w_i - z_j) = 0$. In fact, using basic facts about linear fractional transformations one can show that Graces Apolarity Theorem is equivalent to the statment that $\mathrm{per}(w_i + z_j)$ is stable in the w_i and z_j. A new proof of this follows from the MCP Theorem; from the MCP we get that $\mathrm{per}(w_i + z_j)$ is stable in the z_j if $w_i \in \mathbb{R}$ for all i. Results of Bränden and Borcea then imply it is stable in $w_1, \ldots, w_n, z_1, \ldots, z_n$.

An interesting special case of the MCP is when $c_{ij} = p_i q_j$, where the p_i are real numbers and the q_j are nonnegative reals. Here the MCP reduces to the statement that if the polynomials

$$\sum_{k=0}^{n} d_k z^k, \qquad \sum_{k=0}^{n} m_k z^k.$$

have only real zeros, with the roots of one of them all of the same sign, then

$$\sum_{k=0}^{n} d_k m_k k!(n-k)! z^k$$

also has only real zeros, a result Szegö derived from Grace's Theorem.