The \((q, t)\)-Catalan Numbers

and the

Space of Diagonal Harmonics

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- The $m$-parameter $q, t$-Catalan
In combinatorics a statistic on a finite set $S$ is a mapping from $S \to \mathbb{N}$ given by an explicit combinatorial rule.

**Ex.** Given $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, define

$$\text{inv} \pi = |\{(i, j) : i < j \text{ and } \pi_i > \pi_j\}|$$

and

$$\text{maj} \pi = \sum_{\pi_i > \pi_{i+1}} i.$$

If $\pi = 31542$,

$$\text{inv} \pi = 2 + 2 + 1 = 5$$

and

$$\text{maj} \pi = 1 + 3 + 4 = 8.$$
Let
\[(n)_q = \frac{(1 - q^n)}{(1 - q)} \]
\[= 1 + q + \ldots + q^{n-1}\]
and
\[(n!)_q = \prod_{i=1}^{n} (i)_q \]
\[= (1+q)(1+q+q^2) \cdots (1+q+\ldots+q^{n-1})\]
be the $q$-analogues of $n$ and $n!$.
Then
\[\sum_{\pi \in S_n} q^{\text{inv}\pi} = (n!)_q = \sum_{\pi \in S_n} q^{\text{maj}\pi}.\]
Partitions and the Gaussian Polynomials

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), \( \lambda_i \in \mathbb{N} \) for \( 1 \leq i \leq n \) be a partition and let \( |\lambda| = \sum_i \lambda_i \). Define

\[
\binom{n}{k}_q = \frac{(n!)_q}{(k!)_q((n-k)!)_q}.
\]

**Theorem.** For \( n, k \in \mathbb{N} \),

\[
\binom{n+k}{k}_q = \sum_{(\lambda_1, \ldots, \lambda_n) \leq (k,k,\ldots,k)} q^{|\lambda|}.
\]

Note: We denote the conjugate partition by \( \lambda' \).

**Example** \( n = k = 2 \). The Ferrers shapes are
The Catalan Numbers

\[ C_n = \frac{1}{n + 1} \binom{2n}{n} \]

Recurrence:

\[ C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k} \]

Over 70 interpretations in Stanley’s Enumerative Combinatorics Volume 2, including
• The number of “standard tableaux” of shape $(n, n)$:

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• The number of “Catalan words”, i.e. multiset permutations of $\{0^n1^n\}$ where in any initial segment, the number of zeros is at least as big as the number of ones.

000111 001011 001101 010011 010101

• The number of “Catalan paths” from $(0,0)$ to $(n,n)$, i.e. lattice paths consisting of N and E steps which never go below the main diagonal.
A Catalan Path
Theorem. (MacMahon)

$$\sum_{\text{Catalan words } \sigma} q^{\text{maj}(\sigma)} = \frac{1}{(n + 1)_q} \binom{2n}{n}_q.$$

The Carlitz-Riordan $q$-Catalan

Let $D_n$ denote the set of Catalan paths, and set

$$C_n(q) = \sum_{\sigma \in D_n} q^{\text{area}(\sigma)}$$

where $\text{area}(\sigma)$ is the number of squares below the path and strictly above the diagonal.

Proposition.

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q) C_{n-k}(q).$$
Symmetric Functions

A symmetric function is a polynomial $f(x_1, x_2, \ldots, x_n)$ which satisfies

$$f(x_{\pi_1}, \ldots, x_{\pi_n}) = f(x_1, \ldots, x_n),$$

i.e. $\pi f = f$, for all $\pi \in S_n$.

Examples

• The monomial symmetric functions $m_\lambda(X)$

$$m_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_3 + x_2 x_1 + x_2 x_3 + x_3 x_1 + x_3 x_2.$$  

• The elementary symmetric functions $e_k(X)$

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3.$$  

• The power-sums $p_k(X) = \sum_i x_i^k$.  

• The Schur functions $s_\lambda(X)$, which are important in the representation theory of the symmetric group:

$$s_\lambda(X) = \sum_{\beta \vdash n} K_{\lambda,\beta} m_\beta(X)$$

where $K_{\lambda,\beta}$ equals the number of ways of filling the Ferrers shape of $\lambda$ with elements of the multiset $\{1^{\beta_1} 2^{\beta_2} \cdots \}$, weakly increasing across rows and strictly increasing down columns. For example $K_{(4,2),(2,2,1,1)} = 3$

$$
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 3 & & \\
1 & 1 & 2 & 3 \\
2 & 4 & & 3 & 4
\end{array}
$$
Selberg’s Integral For $k, a, b \in \mathbb{C}$,

\[
\int_{(0,1)^n} \left| \prod_{1 \leq i < j \leq n} (x_i - x_j) \right|^{2k} dx_1 \cdots dx_n
\]

\[
\prod_{i=1}^{n} x_i^{a-1} (1 - x_i)^{b-1} dx_1 \cdots dx_n
\]

\[
= \prod_{i=1}^{n} \frac{\Gamma(a + (i - 1)k)\Gamma(b + (i - 1)k)}{\Gamma(a + b + (n + i - 2)k)} \times \frac{\Gamma(ik + 1)}{\Gamma(k + 1)}.
\]
Macdonald’s Generalization: There exist symmetric functions $P_\lambda(X; q, t)$ such that

$$
\frac{1}{n!} \int_{(0,1)^n} P_\lambda(X; q, t)
$$

$$
\prod_{1 \leq i < j \leq n} \prod_{r=0}^{k-1} (x_i - q^r x_j) (x_i - q^{-r} x_j)
$$

$$
\prod_{i=1}^{n} x_i^{a-1} (x_i; q)_{b-1} d_q x_1 \cdots d_q x_n
$$

$$
= q^F \prod_{i=1}^{n} \frac{\Gamma_q(\lambda_i + a + (i-1)k)}{\Gamma_q(\lambda_i + a + b + (n+i-2)k)} \times \Gamma_q(b + (i-1)k)
$$

$$
\times \prod_{1 \leq i < j \leq n} \frac{\Gamma_q(\lambda_i - \lambda_j + (j-i+1)k)}{\Gamma_q(\lambda_i - \lambda_j + (j-i)k)}
$$
where \( k \in \mathbb{N}, \)

\[
F = k \eta(\lambda) \\
+ k an(n-1)/2 + k^2 n(n-1)(n-2)/3,
\]

\( t = q^k, \)

\[
\Gamma_q(z) = (1-q)^{1-z}(q; q)_\infty/(q^z; q)_\infty
\]

is the \( q \)-gamma function with

\[
(x; q)_\infty = \prod_{i \geq 0} (1 - x q^i),
\]

and

\[
\int_0^1 f(x) d_q x = \sum_{i=0}^{\infty} f(q^i)(q^i - q^{i+1})
\]

is the \( q \)-integral.
**Plethysm:** If $F(X)$ is a symmetric function, then $F[(1 - t)X]$ is defined by expressing $F(X)$ as a polynomial in the $p_k(X) = \sum x_i^k$’s and then replacing each $p_k(X)$ by $(1 - t^k)p_k(X)$.

Macdonald expanded scalar multiples of his $P_\lambda(q, t)$ in terms of the basis $s_\lambda[(1 - t)X]$ and called the coefficients $K_{\lambda,\mu}(q, t)$. He conjectured these coefficients were in $\mathbb{N}[q, t]$. He proved $K_{\lambda,\mu}(1, 1) = K_{\lambda,1}n$ and asked if

$$K_{\lambda,\mu}(q, t) = \sum_T q^{a(\mu,T)} t^{b(\mu,T)}$$

for some statistics $a, b$ on partitions $\mu$ and standard tableaux $T$. 
$$\Delta(\mu) = \begin{vmatrix} 1 & y_1 & x_1 & x_1y_1 & x_1^2 \\ 1 & y_2 & x_2 & x_2y_2 & x_2^2 \\ 1 & y_3 & x_3 & x_3y_3 & x_3^2 \\ 1 & y_4 & x_4 & x_4y_4 & x_4^2 \\ 1 & y_5 & x_5 & x_5y_5 & x_5^2 \end{vmatrix}$$
For $\mu \vdash n$ let $V(\mu)$ denote the linear span over $\mathbb{Q}$ of all partial derivatives of all orders of $\Delta(\mu)$. 
\[ \pi( \text{ a basis element } ) = \text{ linear combo. of basis elements.} \]

\[ \pi( \text{ a basis } ) = \text{ matrix } M(\pi). \]

\[ M(\pi) \ast M(\beta) = M(\pi \ast \beta). \]

The character \( \chi \) is the trace of \( M(\pi) \), which is independent of the basis. Furthermore \( \exists \) a basis for which
\[ M = \begin{bmatrix}
\ast\ast\ast\ast\ast\ast & 0 & 0 \\
0 & \ast\ast\ast & 0 \\
0 & \ast\ast\ast & \ast
\end{bmatrix} \]
$V(\mu)$ decomposes as a direct sum of its bihomogeneous subspaces $V^{i,j}(\mu)$ of degree $i$ in the $x$-variables and $j$ in the $y$-variables. There is an $S_n$-action on $V^{i,j}(\mu)$ given by

$$\pi f = f(x_{\pi_1}, \ldots, x_{\pi_n}, y_{\pi_1}, \ldots, y_{\pi_n})$$

called the diagonal action. The Frobenius Series is the symmetric function

$$\sum_{\lambda \vdash n} s_\lambda(X) \sum_{i,j \geq 0} q^i t^j m_{i,j},$$

where $m_{i,j}$ is the multiplicity of the irreducible $S_n$-character $\chi^\lambda$ in the diagonal action on $V^{i,j}(\mu)$. 
Conjecture. (Garsia, Haiman; PNAS 1993) The Frobenius Series of $V(\mu)$ is given by the modified Macdonald polynomial

$$\tilde{H}_\mu(X; q, t) = \sum_{\lambda \vdash n} t^{\eta(\mu)} K_{\lambda, \mu}(q, 1/t) s_\lambda(X),$$

where $\eta(\mu) = \sum_i (i - 1) \lambda_i$.

Garsia and Haiman also pioneered the study of the space of diagonal harmonics $\mathcal{R}_n$, which is

$$\{ f : \sum_{i=1}^n \partial x_i^h \partial y_i^k f = 0, \forall h+k > 0 \}.$$ 

This is known to be isomorphic to the quotient ring

$$\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/I,$$
where $I$ is the ideal generated by the set of all “polarized power sums” 
$\sum_{i=1}^{n} x_i^h y_i^k, \forall h + k > 0$. The $V(\mu)$ are $S_n$-submodules of $R_n$.

**Conjecture.** (Haiman) The dimension of the space of diagonal harmonics, as a vector space over $\mathbb{Q}$, is $(n + 1)^{n-1}$.

The space $R_n$ decomposes as a direct sum of subspaces of bihomogeneous degree $(i, j)$; $R_n = \bigoplus_{i,j} R_{i,j}^n$. The *Hilbert Series* is the sum

$$\sum_{i,j \geq 0} q^i t^j \dim(R_{i,j}^n).$$

**Example:** If $n = 2$, a basis for
the space is \(1, x_2 - x_1, y_2 - y_1\), and the Hilbert Series is \(1 + q + t\).

The Frobenius Series is the sum

\[
\sum_{\lambda \vdash n} s_\lambda(X) \sum_{i,j \geq 0} q^i t^j m_{i,j}
\]

where \(m_{i,j}\) is the multiplicity of \(\chi^\lambda\) in the character of \(R_{i,j}^n\) under the diagonal action of \(S_n\). For \(n = 2\) this is

\[
s_2(X) + s_{12}(X)(q + t).
\]

Let \(\nabla\) be a linear operator on the basis \(\tilde{H}_\mu(X; q, t)\) given by

\[
\nabla \tilde{H}_\mu(X; q, t) = t^{\eta(\mu)} q^{\eta(\mu')} \tilde{H}_\mu(X; q, t).
\]
Conjecture. (Garsia, Haiman)

The Frobenius Series of $R_n$ is given by $\nabla e_n(X)$.

A polynomial $f$ is alternating if $\pi f = (-1)^{\text{inv}\pi} f$ for all $\pi \in S_n$. A special case of the above conjecture is that the coefficient of $S_{1n}(X)$ in $\nabla e_n(X)$, corresponding to the “sign” character $\chi^{1n}$, is the Hilbert Series of the subspace $R^\varepsilon_n$ of alternates. When $q, t \to 1$ in $\nabla e_n(X)$ they showed this coefficient equals the $n$th Catalan number, which would then equal $\dim(R^\varepsilon_n)$. By results of Macdonald, this coefficient has an expression as a rational function in $q, t$. 
Definition. \((q, t\text{-Catalan})\) Let

\[
C_n(q, t) = (1-q)(1-t) \sum_{\mu \vdash n} t^{2\eta(\mu)} q^{2\eta(\mu')}
\]

\[
\times \frac{\prod' (1 - q^{a'} t^{l'}) \sum q^{a'} t^{l'}}{\prod (q^a - t^{l+1})(t^{l} - q^{a+1})},
\]

where the products are over the squares of \(\mu\), and the arm \(a\), coarm \(a'\), leg \(l\), and coleg \(l'\) of a square are as below.
Conjecture. (Garsia, Haiman; 1992) $C_n(q, t)$ is a polynomial in $q$ and $t$ with nonnegative coefficients.

For $n = 2$ the terms in $C_2(q, t)$ are:

$\mu = 2; \quad \frac{q^2(1 - t)(1 - q)(1 - q)(1 + q)}{(1 - q^2)(q - t)(1 - q)(1 - t)}$

$\mu = 1^2; \quad \frac{t^2(1 - t)(1 - q)(1 - t)(1 + t)}{(1 - t^2)(t - q)(1 - t)(1 - q)}$

So

$$C_2(q, t) = \frac{t^2}{t - q} + \frac{q^2}{q - t} = \frac{t^2 - q^2}{t - q} = t + q.$$
After simplification the terms in $C_3(q, t)$ are

$$\mu = 3; \quad \frac{q^6}{q^2 - t}$$

$$\mu = 21; \quad \frac{t^2 q^2 (1 + q + t)}{(q - t^2) (t - q^2)}$$

$$\mu = 1^3; \quad \frac{t^6}{(t^2 - q)(t - q)}$$

So

$$C_3(q, t) =$$

$$\frac{q^6 (t^2 - q) + t^2 q^2 (1 + q + t) (q - t) + t^6 (t - q^2)}{(q^2 - t) (t^2 - q)(q - t)}$$

$$= q^3 + q^2 t + qt^2 + qt + t^3.$$
Theorem. (Garsia, Haiman)

\[ q^{\binom{n}{2}} C_n(q, 1/q) = \frac{1}{(n + 1)_q} \binom{2n}{n}_q. \]

Theorem. (Garsia, Haiman)

\[ C_n(q, 1) = \sum_{\sigma \in D_n} q^{\text{area}(\sigma)}. \]

Problem: Is there a pair of statistics \((q\text{stat}, t\text{stat})\) on Catalan paths such that

\[ C_n(q, t) = \sum_{\sigma \in D_n} q^{q\text{stat}(\sigma)} t^{t\text{stat}(\sigma)}. \]
**Theorem.** (Haiman; JAMS 2001) If $\mu \vdash n$, the Frobenius Series of $V(\mu)$ is the modified Macdonald polynomial $\tilde{H}_\mu(X; q, t)$.

**Pf:** Algebraic Geometry and Commutative Algebra.

**Corollaries.** For all $\lambda, \mu \vdash n$,

$$K_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t] \text{ and } \dim(V(\mu)) = n!.$$

So far no pair of statistics for the $K_{\lambda, \mu}(q, t)$ have been proposed.

**Theorem.** (Garsia, H.; PNAS 2001)

$$C_n(q, t) \in \mathbb{N}[q, t].$$

**Pf:** Intricate application of plethystic identities involving $\nabla$ after an empirical discovery of a recurrence.
The circles form the bounce path.
The bounce statistic is $2 + 4 + 7 = 13$. 
Definition.

\[ F_n(q, t) = \sum_{\sigma \in \mathcal{D}_n} q^{\text{area}(\sigma)} t^{\text{bounce}(\sigma)}. \]

Conjecture. (H.; To appear in Adv. in Math.) For all \( n \in \mathbb{N} \),

\[ F_n(q, t) = C_n(q, t). \]

(Verified in Maple for \( n \leq 14 \)).
Definition. Say \( \sigma \) ends in \( \text{end}(\sigma) \) \( E \) steps. For \( n, s \in \mathbb{N} \), set
\[
F_{n,s}(q,t) = \sum_{\sigma \in \mathcal{D}_n, \text{end}(\sigma) = s} q^{\text{area}(\sigma)} t^{\text{bounce}(\sigma)}.
\]

Theorem.
\[
F_{n,s}(q,t) = \sum_{r=0}^{n-s} q^{\left(\frac{s}{2}\right)} t^{n-s} \times F_{n-s,r}(q,t) \binom{r + s - 1}{r}_q.
\]

Corollary.
\[
q^{\binom{n}{2}} F_n(q, q^{-1}) = \frac{1}{(n + 1)_q} \binom{2n}{n}_q.
\]
Theorem. (Garsia, H.; PNAS 2001) For all $n, s \in \mathbb{N}$,

$$t^{n-s} q^{\binom{s}{2}} \nabla e_{n-s} [X \frac{1 - q^s}{1 - q}] |_{s_1 n - s}(X)$$

$$= F_{n,s}(q,t).$$

Corollary.

$$C_n(q,t) = F_n(q,t).$$

Corollary.

$$F_{n,s} = (1 - q^s) \sum_{\mu \vdash n} t^{\eta(\mu)} q^{\eta(\mu')}$$

$$\times \frac{\prod' (1 - q^{a'} t^{l'}) h_s [(1 - t) \sum q^{a'} t^{l'}]}{\prod (q^a - t^{l+1})(t^l - q^{a+1})}.$$
Haiman discovered another pair of statistics for the $q, t$-Catalan.

**Conjecture.** (Haiman)

\[ C_n(q, t) = \sum_{\sigma \in \mathcal{D}_n} q^{\text{area}(\sigma)} t^{\text{dinv}(\sigma)}. \]

**Proposition.**

\[ \sum_{\sigma \in \mathcal{D}_n} q^{\text{area}(\sigma)} t^{\text{dinv}(\sigma)} = \sum_{\sigma \in \mathcal{D}_n} q^{\text{bounce}(\sigma)} t^{\text{area}(\sigma)}. \]

**Corollary.** Haiman’s conjecture above is true.
The statistic $d_{inv}$ is the \# of pairs $(i, j), i < j$ with the lengths $r_i$ and $r_j$ of rows $i, j$ satisfying $r_j - r_i \in \{0, 1\}$. 
Corollary. $F_n(q, 1) = F_n(1, q)$.

Open Question. Find a bijective proof that $F_n(q, t) = F_n(t, q)$.

Theorem. (Haiman; Invent. Math. 2002) $\nabla e_n(X)$ is the Frobenius Series of $R_n$.

Corollary. The $(q, t)$-Catalan $C_n(q, t)$ is the Hilbert Series of the space of alternates $R_n^e$.

Corollary. $\dim(R_n) = (n+1)^{n-1}$.

The number $(n+1)^{n-1}$ is the number of rooted, labeled trees on $n+1$ vertices, with root node labeled 0, and also the number of parking functions on $n$ cars.
\[ \text{dinv} = \#(i, j), i < j : r_i = r_j \text{ and } \text{car}_i > \text{car}_j \text{ or } r_i = r_j - 1 \text{ and } \text{car}_i < \text{car}_j. \]
**Conjecture.** (H., Loehr) The Hilbert Series of $R_n$ is given by

$$W_n(q, t) = \sum_{\sigma} q^{\text{area}(\sigma)} t^{\text{dinv}(\sigma)},$$

where the sum is over all parking functions on $n$ cars.

Using Maple, we have verified our conjecture for $n \leq 7$. We can’t prove, by any method, that $W_n(q, t) = W_n(t, q)$, nor can we prove that

$$q^{\binom{n}{2}} W_n(q, 1/q) = (1 + q + \ldots + q^n)^{n-1},$$

which is the value for the Hilbert Series at $t = 1/q$ conjectured by Stanley and now proven by Haiman. Loehr has a proof that $W_n(q, 1) = W_n(1, q)$. 
Garsia and Haiman define
\[ C_n^m(q, t) = \nabla^m e_n(X)|_{s_1 n}(x), \quad m \in \mathbb{N}. \]
Note \( C_n^1(q, t) = C_n(q, t) \). These are connected to lattice paths from \((0, 0)\) to \((nm, n)\) which never go below the diagonal, and also have an algebraic description.

**Conjecture.** (Haiman, Loehr)

\[
\sum_{\sigma \in \mathcal{D}_n^m} q^{\text{area}(\sigma)} t^{m-\text{dinv}(\sigma)} = C_n^m(q, t)
\]

\[
= \sum_{\sigma \in \mathcal{D}_n^m} q^{\text{area}(\sigma)} t^{m-\text{bounce}(\sigma)}.
\]

Loehr obtains recurrences involving the parameter \( m \) which extend the recurrence for \( F_{n,s}(q, t) \).
Lapointe, Lascoux and Morse have introduced a generalization of Schur functions they call “Atoms”, which depend on $X$, $t$, a positive integer $k$, and a partition $\lambda$ satisfying $\lambda_1 \leq k$. The coefficients in the expansion of the Atoms in terms of Schur functions are in $\mathbb{N}[t]$, and they conjecture that if $\mu_1 \leq k$, the coefficients in the expansion of the $\tilde{H}_\mu(X; q, t)$ in terms of the Atoms are in $\mathbb{N}[q, t]$. This conjecture thus implies $K_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$. Hear more about this in the special session on Algebraic and Enumerative Combinatorics.
The bounce path for the case $m = 2$. Go up distance $a_1$ to the path, then over $a_1$, then up distance $a_2$, then over $a_1 + a_2$, then up $a_3$, then over $a_2 + a_3$, etc.
Start with the path above. Form the bounce path (circles, next page) whose top step is the # of rows length zero, etc. Then start at corner of top step, and look at subword of 0’s and 1’s on previ-
(area, dinv) → (bounce, area)

Ours page, starting at bottom. For each 0 go down, for each 1 go left. Then iterate with subword of 1’s and 2’s.