From Connections to Relationships with Cellular Sheaves

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CELLULAR SHEAVES

LOCAL \rightarrow GLOBAL

local relationships define global structure for data on a network
local relationships

1

data here is equal to data here

−1

data here is opposite to data here

global structure

constant functions on vertices

more complicated behavior
A sheaf $\mathcal{F}$ over a graph $G$ is given by:

<table>
<thead>
<tr>
<th>stalks</th>
<th>restriction maps</th>
</tr>
</thead>
<tbody>
<tr>
<td>A vector space $\mathcal{F}(v)$ for each vertex $v$ of $G$</td>
<td>A linear map $\mathcal{F}_{v \leq e} : \mathcal{F}(v) \to \mathcal{F}(e)$ for each incident vertex-edge pair $v \leq e$ of $G$</td>
</tr>
<tr>
<td>A vector space $\mathcal{F}(e)$ for each edge $e$ of $G$</td>
<td></td>
</tr>
<tr>
<td>hold local data on the graph $x_v \in \mathcal{F}(v)$</td>
<td>specify local consistency conditions for data $\mathcal{F}<em>{u \leq e} x_u = \mathcal{F}</em>{v \leq e} x_v$</td>
</tr>
</tbody>
</table>

**Sections:** Global assignments to stalks that satisfy all local consistency conditions

**Cochains:** Global assignments to stalks with no conditions

**CONSTANT SIGNALS**

$\mathcal{F}(u) \xrightarrow{\mathcal{F}_{u \leq e}} \mathcal{F}(e) \xleftarrow{\mathcal{F}_{v \leq e}} \mathcal{F}(v)$

**SIGNALS**

$u \xrightarrow{e} v$
What is a section?

Sections lift the underlying graph into the stalks.
What is a section?

Sections are hard to find—there may not be any!
What is a cochain?

Data associated to a network usually comes in the form of cochains.
Working with Cochains

**COBOUNDARY MAP**

Computes discrepancy of a signal over edges

\[(\delta x)_e = F_{u \subseteq e} x_u - F_{v \subseteq e} x_v\]

Analogous to incidence matrix

\[\ker \delta = \text{space of global sections}\]

**SHEAF LAPLACIAN**

\[L = \delta^T \delta\]

\[(Lx)_v = \sum_{v \subseteq e} F^T_{v \subseteq e} (F_{v \subseteq e} x_v - F_{u \subseteq e} x_u)\]

\[\ker L = \text{space of global sections}\]

**Energy of a cochain**

\[\mathcal{E}(x) = x^T Lx = \sum_{e = u \sim v} \| F_{v \subseteq e} x_v - F_{u \subseteq e} x_u \|^2\]

Measures distance from being a section
Some Classes of Sheaves

**Constant sheaf**
- Stalks: \( \mathbb{R} \)
- Restriction maps: 1

**Signed graph**
- Stalks: \( \mathbb{R} \)
- Restriction maps: ±1

**Connection graph**
- Stalks: \( \mathbb{R}^n \)
- Restriction maps in \( O(n) \)

\[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\]

**Matrix-weighted graph**
- Stalks: \( \mathbb{R}^n \)
- SPD weight matrices on edges

\[
\begin{pmatrix}
I_n & -\rho \\
-\rho^T & I_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
W & -W \\
-W & W
\end{pmatrix}
\]
Connections with Multilayer Networks

Layers of the network become dimensions of stalks
Restriction maps identify intra-layer connections
Sheaves can encode algebraic mixing between layers
Dynamics on Sheaves

DIFFUSION

\[ \dot{x} = -\alpha L x \]

Essentially gradient descent on \( \mathcal{E}(x) \)
Trajectories converge to global sections
Distributed dynamics

Use as a component for more complicated network dynamics
(react-ion-diffusion)

EXAMPLES

Simple extensions of opinion dynamics:
• High-dimensional opinion spaces
• Friend/foe relationships
• Translation/misinterpretation of opinions

Multiplex networks:
• Sheaf diffusion implements intra-layer dynamics
• Other independent processes can give inter-layer interactions
Learning Sheaves

**Given:** signals $x_i$ supported on nodes of an unknown network

**Learn:** which edges should be present in the network

a sheaf where the signals are close to being sections

Sheaf Laplacians form a convex cone. Minimize

$$\sum_{i=1}^{n} \mathcal{E}(x_i) = \text{tr}(X^T LX)$$

over this cone, with some regularization terms.

The learned sheaf Laplacian expresses the tendency of observed data over one node to have a linear relationship with data over its neighbors.
Wrapping Up and Moving Forward

More topological and spectral tools for sheaves:
Pushforwards and pullbacks for network decomposition
Sheaves on complexes for higher-dimensional relationships

Sheaves should admit versions of graph-based network concepts like modularity/clustering, assortativity, centrality

Seeking network data that might be better explained by a sheaf, networks that might benefit from sheaf modeling