Level Set Determines Sub-Level Set Persistence

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Abstract

Here we leverage the sheaf theoretic approach to show how level set persistence determines sublevel set persistence. This result holds in the multi-dimensional setting as well.

1 Setup for the 1D Case

Suppose $f : X \to \mathbb{R}$ is a proper stratified map. This means that there is a decomposition of $\mathbb{R}$ into open cells so that over each interval (x_n, x_{n+1}), the map f restricts to a fiber bundle.

Using Lemma 5.4.3 of *Sheaves, Cosheaves and Application* (SCA), for each k we can define a cellular cosheaf $\hat{F}_k$ - called the kth Leray cosheaf - that assigns to each open interval $a_n := (x_n, x_{n+1})$ the vector space

$$H_k(f^{-1}(t_n); k)$$

where $t_n \in (x_n, x_{n+1})$. Over the vertices x_n, one assigns the vector space $H_k(f^{-1}(x_n); k)$. The extension maps

$$\hat{F}_k(a_n) \to \hat{F}_k(x_n)$$

are defined using a suitable regular neighborhood of the fiber $f^{-1}(x_n)$.

Corollary 8.6 of SCA tells us that these cosheaves participate in a Leray spectral sequence that terminates at the E^2 page (because $\mathbb{R}$ is one-dimensional). In particular the homology of X can be computed as follows.

$$H_k(X; k) \cong H_0(\mathbb{R}; \hat{F}_k) \oplus H_1(\mathbb{R}; \hat{F}_{k-1}) \cong H_0^{BM}(B_k) \oplus H_1^{BM}(B_{k-1})$$

Here B_k refers to the barcode decomposition of the cosheaf $\hat{F}_k$. The superscript BM refers to Borel-Moore homology of the barcode viewed as a space. Borel-Moore homology can be thought of as the linear dual of compactly-supported cohomology. The second isomorphism tells us that in $1d$ we can compute cosheaf homology via Borel-Moore homology of the barcodes. The composition of these isomorphisms was first observed by Tamal Dey and Dan Burghalea, but they did not use cosheaves.

Observe, that since $f : X \to \mathbb{R}$ is a proper stratified map, the restriction of f to $X_{\leqslant t} := f^{-1}((-\infty, t])$ is also a proper stratified map. Consequently, we can apply the above result

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to the space $X_{\leqslant t}$ instead, but we have to restrict the cosheaves to the subspace $(-\infty, t]$. Fortunately, restriction of a sheaf or cosheaf to a subspace is a standard operation in the Grothendieck six-functor formalism: If $\iota : (-\infty, t] \hookrightarrow \mathbb{R}$ is the inclusion, then the application of the above result to the restriction reads

$$H_k(X_{\leqslant t}; k) \cong H_0((-\infty, t]; \iota^* \hat{F}_k) \oplus H_1((-\infty, t]; \iota^* \hat{F}_{k-1})$$

The upshot of this formula is that we can define a family of vector spaces, one for each $t \in \mathbb{R}$ that records the homology of the sublevel set $X_{\leqslant t}$

$$S(t) := H_0((-\infty, t]; \iota^* \hat{F}_k) \oplus H_1((-\infty, t]; \iota^* \hat{F}_{k-1})$$

given by computing cosheaf homology of the restriction of the Leray cosheaves to the subspace $(-\infty, t]$. What remains to be shown is that there are maps

$$S(t) \to S(t') \qquad t \leqslant t'$$

that can be defined purely cosheaf-theoretically. To do this, we will make use of some standard adjunctions in (co)sheaf theory.

2 The Proof Using Sheaves

Most of the proof can dualize back the language of cosheaves, but the result is most reliably proven using the language of sheaves, so we will work with sheaves and dualize back at the end to cosheaves. The reason is that there is one result that I am not sure dualizes readily to cosheaves: if j is the inclusion of a closed subspace, then j_* is exact. First, let us review an elementary adjunction.

Theorem 2.1 (Thm 6.20 of SCA). *Let* $f : Y \to Z$ *be a continuous map. The functors* $f^* : \mathbf{Shv}(Z) \to \mathbf{Shv}(Y)$ *and* $f_* : \mathbf{Shv}(Y) \to \mathbf{Shv}(Z)$ *form an adjoint pair* (f^*, f_*) *and thus*

$$\mathbf{Hom}_{\mathbf{Shv}(Y)}(f^*G, F) \cong \mathbf{Hom}_{\mathbf{Shv}(Z)}(G, f_*F).$$

Dually, the functors for cosheaves satisfy the opposite adjunction (f_*, f^*)

$$\mathbf{Hom}_{\mathbf{CoShv}(Z)}(f_* \hat{F}, \hat{G}) \cong \mathbf{Hom}_{\mathbf{CoShv}(Y)}(\hat{F}, f^* \hat{G}).$$

In the above adjunction for sheaves, let $Y = (-\infty, t]$, $Z = (-\infty, t']$ and $f = j$ be the inclusion of Y as a closed subspace of Z. Observe that if we set $F = j^*G$ in the above adjunction, then we get an isomorphism

$$\mathbf{Hom}_{\mathbf{Shv}(Y)}(j^*G, j^*G) \cong \mathbf{Hom}_{\mathbf{Shv}(Z)}(G, j_*j^*G).$$

that is natural in G. This defines the unit of the adjunction:

$$\mathrm{id}_{\mathbf{Shv}(Y)} \to j_*j^*.$$

Proposition 2.2 (Iversen II.5 p. 102). *For $j : Y \hookrightarrow Z$ the inclusion of a closed subspace, the functor $j_* : \mathbf{Shv}(Y) \to \mathbf{Shv}(Z)$ is exact, i.e. it sends exact sequences of sheaves to exact sequences of sheaves. Moreover j^* is always exact.*

Lemma 2.3. *Suppose $j : Y \hookrightarrow Z$ is the inclusion of a closed subspace and F is a sheaf on Z, then there is an induced map from the cohomology of F on Z to the cohomology of j^*F on Y.*

$$H^i(Z; F) \to H^i(Y; j^*F)$$

Proof. We can read off the proof from the following diagram of spaces.

Sheaf cohomology is defined as the right derived functor of pushforward to a point. If we want to compute sheaf cohomology of F, one takes an injective resolution of F

$$0 \to F \to I^\bullet$$

and applies p_{Z*} to the injective resolution.

$$Rp_{Z*}F := p_{Z*}I^\bullet$$

This results in a chain complex of vector spaces, whose cohomology is the sheaf cohomology of F. We usually save this step for last as it takes us out of the category of chain complexes of vector spaces and into the category of graded vector spaces. This is written as follows.

$$R^i p_{Z*}F := H^i(p_{Z*}I^\bullet) =: H^i(Z; F)$$

Since j^* is exact we will consider an injective resolution of F and pull that back to an injective resolution of j^*F. Observe the following string of identities.

$$
\begin{aligned}
Rp_{Y*}j^*F &:= p_{Y*}j^*I^\bullet \\
&= (p_Z \circ j)_* j^*I^\bullet \\
&= p_{Z*}j_* j^*I^\bullet
\end{aligned}
$$

The unit of the adjunction defines a map of sheaves

$$F \to j_* j^*F,$$

which also defines a map on complexes of sheaves and hence injective resolutions.

$$I^\bullet \to j_* j^*I^\bullet$$

Because j^* and j_* are exact and preserve injectives, $j_*j^*I^\bullet$ is an injective resolution of j_*j^*F, thus
$$Rp_{Z*}j_*j^*F = p_{Z*}j_*j^*I^\bullet = Rp_{Y*}j^*F$$
and hence the unit of the adjunction defines a map
$$Rp_{Z*}F \to Rp_{Y*}j^*F \quad \Rightarrow \quad H^i(Z;F) \to H^i(Y;j^*F.)$$

$\square$

Remark 2.4 (Abuse of Notation). Common practice in the sheaf literature is to suppress the notation j^*F and to just write
$$H^i(Y;F) := H^i(Y;j^*F).$$

The reasoning is that F is a sheaf on Z and hence the only way to parse the formula on the left is to realize that the sheaf must be restricted to the subspace Y.

As a corollary we obtain our desired result.

Theorem 2.5 (Main Result in 1D). *If $f : X \to \mathbb{R}$ is a proper map and F^k is the Leray sheaf whose stalk at x is the cohomology of the fiber $H^k(f^{-1}(x))$, then we can define a functor*

$$S^k : \mathbb{R}^{op} \to \mathbf{Vect} \qquad S^k(t) := H^0((-\infty, t]; F^k) \oplus H^1((-\infty, t]; F^{k-1}) \cong H^k(f^{-1}(-\infty, t])$$

whose value records the cohomology of the entire sublevel set. The maps

$$S^k(t') \to S^k(t) \qquad t \leqslant t'$$

are defined sheaf theoretically by observing that we have maps

$$H^0((-\infty, t']; F^k) \to H^0((-\infty, t]; F^k) \qquad H^1((-\infty, t']; F^{k-1}) \to H^1((-\infty, t]; F^{k-1})$$

the sum of which define the desired map.

3   Comments on the Multi-D Setting

The result for base spaces of higher dimension is more complicated, because the sublevel set cohomology is not simply a direct sum of two sheaf cohomology groups. There are higher Leray differentials that weave together the results in a more subtle way. Nevertheless, the answer exists and uses hypercohomology.

To be completed...