DIRECTIONS: Part A has 4 shorter problems (5 points each) while Part B has 6 traditional problems (10 points each). To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Four shorter Problems, 5 points each.

A-1. Determine the image of the function \( f(x) := \frac{2x^2}{1 + x^2} \).

A-2. If \( a \) and \( b \) are rational numbers, consider the set \( S \) of real numbers of the form \( a + b\sqrt{7} \). Show that the elements in \( S \) have multiplicative inverses in \( S \). [This is the key step in showing that \( S \) is a field.]

A-3. Determine if the set \( S = \{ x \in \mathbb{R} : 4x^2 > x^3 + 3x \} \) is bounded above and/or below, and if so, find \( \inf(S) \) and \( \sup(S) \) – if they exist.

A-4. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be bounded functions such that \( f(x) \leq g(x) \) for all \( x \). Let \( F \) denote the image of \( f \) and \( G \) the image of \( g \). Give an example (a picture) of pairs of such functions with \( \sup(F) > \inf(G) \).
Part B: Six traditional problems, 10 points each.

B-1. Let $n$ be a positive integer. For any integers $a, b$ we say that $a$ equals $b \mod n$ if $a$ and $b$ have the same remainders when divided by $n$ (equivalently, if $b - a$ is divisible by $n$). We write: $a \equiv b \pmod{n}$. So modulo $n$ the possible remainders are $0, 1, 2, \ldots, (n - 1)$ and every integer is equivalent to one of these.

If $a \equiv r \pmod{n}$ and $b \equiv s \pmod{n}$, show that $ab \equiv rs \pmod{n}$.

As a special case, since $3^4 \equiv 1 \pmod{5}$, then $3^8 \equiv? \pmod{5}$ and $3^9 \equiv? \pmod{5}$.

B-2. If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 4}$ for $n \geq 1$, show that $a_n < 4$ for all $n \geq 1$. 
B-3. Let $c$ be a complex number with $|c| < 1$. Show that $2nc^n \to 0$.

B-4. For each condition below, give an example of an unbounded sequence such that $a_{n+1} - a_n > 0$ for all $n \in \mathbb{N}$ and the specified condition holds.

a) $\lim (a_{n+1} - a_n) = L$, where $L > 0$.

b) $\lim (a_{n+1} - a_n) = 0$. 
B-5. Let $a_n$ and $b_n$ be sequences of real numbers. If $a_n \to A$ and $b_n \to B$, show that $a_nb_n$ converges to $AB$.

B-6. Let $b_n$ be a sequence of real numbers with the properties $b_n > 0$ and $b_n \to B$. Show that $B \geq 0$.

Also, give an example where $B = 0$. 