The problem numbers refer to the D’Angelo-West text.

For the True-False [T/F] questions, either provide a proof or give a counterexample.

1. [#15.13] Use \( \epsilon \)-\( \delta \) to show that the function \( |x| \) is continuous for all \( x \in \mathbb{R} \).

   **Solution:** Use three cases, Case 1: At a point \( x = a > 0 \). Then this is the function \( x \), which we know is continuous – as long as we choose a small enough interval about \( a \) in which \( x > 0 \). For this choose \( \delta \) so small that in the interval \( |x - a| < \delta \) both \( x > 0 \) and \( |x - a| < \epsilon \). The first condition is satisfied if, say, \( \delta < a/2 \) since then \( x > a/2 \).

   The second condition is satisfied if \( \delta \leq \epsilon \). If both of these are satisfied, then \( |x| = x \) so that \( |x - a| = |x - a| < \epsilon \).

   Case 2: At a point \( x = a < 0 \). Since \( |-x| = |x| \), the functions \( |x| \) and \( |-x| \) are identical so we can reduce to Case 1.

   Case 3: At the point \( x = 0 \), pick \( \delta = \epsilon \). Then, if \( |x| < \delta \), we also have \( |x - a| < \epsilon \).

2. [#15.2] [T/F] There is a continuous \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(x) = 0 \) if and only if \( x \) is an integer.

   **Solution:** True. \( f(x) = \sin(\pi x) \) or \( |\sin \pi x| \) etc. Or just draw a sketch.

3. [#15.3] [T/F] If \( f : \mathbb{R} \to \mathbb{R} \) is continuous everywhere and \( f(x) = 0 \) for all rational numbers \( x \), then \( f(x) = 0 \) for all real \( x \).

   **Solution:** True. For any \( a \in \mathbb{R} \), there is a sequence of rational numbers \( x_n \to a \). Since \( f \) is continuous, \( f(a) = \lim_{n\to\infty} f(x_n) = 0 \).

4. [#15.4] [T/F] There exists \( x > 1 \) such that \( \frac{x^2 + 5}{3 + x^3} = 1 \).

   **Solution:** True. Let \( f(x) = \frac{x^2 + 5}{3 + x^3} \). Since \( f(1) = \frac{6}{4} > 1 \) and since \( f(2) = \frac{9}{13} < 1 \), there is \( x \in (1, 2) \) such that \( f(x) = 1 \).

   **Alternate:** This is equivalent to showing that the polynomial \( p(x) = x^2 + 5 - 3 - x^7 \) has a root at some point \( x > 1 \). Clearly \( p(1) = 2 > 0 \) while \( p(2) = 2^2 + 5 - 3 - 2^7 < 0 \). Now use the intermediate value theorem.

5. [#15.5] [T/F] The function \( f(x) := |x|^3 \) is continuous for all \( x \in \mathbb{R} \).
**Solution:** True. **Method 1:** Write $|x|^3 = x^2|x|$. This is the product of two continuous functions.

**Method 2:** Let $f(x) = x^3$ and $g(x) = |x|$. $f$ is continuous since it is a polynomial. $g$ is continuous by problem 1. So their composition $g \circ f$ is continuous, that is, $|x^3|$ is continuous.

6. [#15.7] [T/F] Let $f, g$, and $h$ be continuous on the interval $[0, 2]$. If $f(0) < g(0) < h(0)$ and $f(2) > g(2) > h(2)$, then there exists some $c \in [0, 2]$ such that $f(c) = g(c) = h(c)$.

**Solution:** False. The easiest way to see this is to try to draw a sketch. You will immediately find a counterexample to the statement.

It is then easy to do this using formulas. For instance, let $f(x) = 2x$, $g(x) = 1$ and $h(x) = 4 - 2x$. Then $f(0) < g(0) < h(0)$ and $f(2) > g(2) > h(2)$. If $f(x) = g(x) = h(x)$ for some $x$, then $2x = 1 = 4 - 2x$, which is impossible.

7. [#15.8] [T/F] Let $f : \mathbb{R} \to \mathbb{R}$. If $|f|$ is continuous, then $f$ is continuous.

**Solution:** False. Define $f(x) = 1$ when $x > 0$ and $f(x) = -1$ when $x \leq 0$, then $|f(x)| = 1$ is continuous but $f(x)$ is not.

8. [#15.10][T/F]

   a) If $f$ is continuous on $\mathbb{R}$, then $f$ is bounded.
   b) If $f$ is continuous on $[0,1]$, then $f$ is bounded.
   c) If $f$ is continuous on $\mathbb{R}$ and is bounded, then $f$ attains its supremum.

**Solution:** (a) False. $f(x) = x$ is continuous on $\mathbb{R}$, but $f$ is unbounded.
   
   (b) True. Theorem 15.24.
   
   (c) False. $f(x) = x^2/(1 + x^2)$ is continuous on $\mathbb{R}$, its supremum is $+1$ but does not attain its supremum.

9. [#15.15] Let $f(x) := x^2 + 4x$. Clearly $\lim_{x \to 0} f(x) = 0$. Assuming that $0 < \epsilon < 4$, how small must $\delta$ be so that $|x| < \delta$ implies that $|f(x)| < \epsilon$? Express $\delta$ as a function of $\epsilon$.

**Solution:** Since $f(x) = x(x+4)$ we put a preliminary restriction on $x$ to control the term $x+4$. For instance, require that $|x| < 1$ so we are requiring that $\delta < 1$. If $|x| < \delta$ then

$$|f(x)| = |x^2 + 4x| \leq |x||x+4| < 5\delta.$$  

To insure that $|f(x)| < \epsilon$ in the interval we also choose $5\delta \leq \epsilon$. Consequently, choose $\delta = \min(1, \frac{\epsilon}{5})$.  

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10. [\#15.12] Construct a function $f$ with the property that there are sequences $a_n$ and $b_n$ converging to zero such that $f(a_n)$ converges to zero but $f(b_n)$ is unbounded.

Does there exist such a function $f$ that is continuous at $x = 0$?

**Solution:** Define $f(x) = 0$ for $x \leq 0$ and $f(x) = \frac{1}{x}$ for $x > 0$. Let $a_n = 0$ and $b_n = \frac{1}{n}$. Then $f(a_n) = 0$ converges to zero and $f(b_n) = n$ is unbounded.

**Another example.** Let $f(x) = \frac{1}{x} \sin \frac{1}{x}$ for $x > 0$. Since $\sin \theta = 0$ at $\theta = n\pi$, $n = 1, 2, \ldots$ and $\sin \theta = 1$ at $\theta = \frac{n\pi}{2} + 2n\pi$, $n = 1, 2, \ldots$, let $a_n = 1/(n\pi)$ and $b_n = 1/(\frac{n\pi}{2} + 2n\pi)$.

When $f$ is continuous, there is no such $f$ since $\lim_{n \to \infty} f(b_n) = f(0)$.

11. [\#15.17] Let $f(a, n) := (1 + a)^n$, where $a$ and $n$ are positive.

a) For constant $a$, how does $f(a, n)$ behave as $n \to \infty$? For constant $n$, how does $f(a, n)$ behave as $a \to 0$?

b) Let $L \geq 1$ be a given real number. Prove that there exists a sequence $a_n \to 0$ and $f(a_n, n) \to L$ as $n \to \infty$. In other words, depending on the choice of $a_n$, $f$ may approach any value.

**Solution:** (a) Since $(1 + a) > 1$, then $(1 + a)^n$ diverges to infinity as $n \to \infty$.

If we fix $n$, $f(a, n)$ is a polynomial of degree $n$, and hence continuous. So $\lim_{a \to 0} f(a, n) = f(0, n) = 1$.

(b) We can pick $a_n$ such that $f(a_n, n) = L$, that is, $(1 + a_n)^n = L$. Solving this we get $a_n = L (\frac{1}{n} - 1)$. Note that with this choice $a_n \to 0$ as $n \to \infty$.

12. Given any real number $c > 0$, prove there is an $x > 0$ such that $x^{17} = c$.

**Solution:** Let $f(x) = x^{17}$, then $f(0) = 0 < c$. $f(1 + c) = (1 + c)^{17} \geq 1 + 17c > c$. Hence there is $x \in (0, 1 + c)$ such that $f(x) = c$.

This is a special case of the fact that any polynomial $p(x)$ whose degree is odd has at least one real zero (show that for large enough $x$, then $p(\pm x)$ have opposite sign. The result then follows from the intermediate value theorem.

13. [\#15.21] Prove that there exists $x \in [1, 2]$ such that $x^5 + 2x + 5 = x^4 + 10$.

**Solution:** This is also an easy consequence of the intermediate value theorem. Let $f(x) = x^5 + 2x + 5 - x^4 - 10$. Then $f(1) = -3 < 0$ and $f(2) = 15 > 0$. So there is $x \in (1, 2)$ such that $f(x) = 0$. In other words, $x^5 + 2x + 5 = x^4 + 10$ holds for this $x$.