The problem numbers refer to the D’Angelo-West text.

1. [#16.1] For \( x \neq 0 \) compute \( \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{(x+h)^2} - \frac{1}{x^2} \right) \).

2. [#16.11] Use the definition of the derivative as the limit of a difference quotient to derive the product rule for differentiating \( f(x)g(x) \). [SUGGESTION: Add and subtract an appropriate quantity in the numerator.]

3. Use the definition of the derivative as the limit of a difference quotient to derive the formula for the derivative of \( f(x) = \sqrt{x} \) for \( x > 0 \).

4. Let a smooth function \( g(x) \) have the three properties: \( g(0) = 3, \ g(1) = 1, \ g(4) = 7 \).
   a) Show that at some point \( 0 < c < 4 \) one has \( g''(c) > 0 \). Better yet, find a number \( m > 0 \) so that \( g''(c) \geq m > 0 \).
   b) Is it true that \( g'' \) must be positive at at least one point in the interval \( 0 < x < 1 \)? Proof or counterexample.
   c) [This is the optimal version of part (a)]. Let \( (x_1, y_1), (x_2, y_2), (x_3, y_3) \) be any three points in the plane with \( x_1 < x_2 < x_3, \ y_1 > y_2, \) and \( y_3 > y_2 \). Then there is a point \( c \in (x_1, x_3) \) such that \( g''(c) = m > 0 \), where \( m \) is the second derivative of the (unique) quadratic polynomial passing through the three points.

5. Let \( v(x) \) be a smooth real-valued function for \( 0 \leq x \leq 1 \). If \( v(0) = v(1) = 0 \) and \( v''(x) > 0 \) for all \( 0 \leq x \leq 1 \), show that \( v(x) \leq 0 \) for all \( 0 \leq x \leq 1 \).

6. Let \( g(x) \) is a smooth function with \( g(2) = 0 \) and let \( f(x) = x^2 g(x) \). Use the mean value theorem to show that \( f''(c) = 0 \) for some \( 0 < c < 2 \).

7. a) Let \( g(x) := x^3(1 - x) \). Use the mean value theorem to show that \( g'''(c) = 0 \) for some \( 0 < c < 1 \).
   b) Let \( h(x) := x^3(1 - x)^3 \). Show that \( h'''(x) \) has exactly three distinct roots in the interval \( 0 < x < 1 \).
c) Let \( p(x) := \left( \frac{d}{dx} \right)^4 (1 - x^2)^4 \). Show that \( p \) is a polynomial of degree 4 and that it has 4 real distinct zeroes, all lying in the interval \(-1 < x < 1\).

8. If \( b \geq 0 \), show that for every real \( c \) the equation \( x^5 + bx + c = 0 \) has exactly one real root.

9. Let \( p(x) := x^3 + 3cx + d \), where \( c \) and \( d \) are real. Under what conditions on \( c \) and \( d \) does this have three distinct real roots? [SUGGESTION: Look at the graph of \( p \) and observe something simple about the local maximum and local minimum for \( p \) to have three distinct real roots.] [ANSWER: \( c < 0 \) and \( d^2 < -4c^3 \)].

10. [#16.31] Let \( f(x) \) be a differentiable function for all real \( x \) with the property that \( f'(x) < 1 \) for all \( x \). Show has at most one fixed point, that is, at most one point \( p \) where \( f(p) = p \).

11. Let \( f(x) \) be a differentiable function for all real \( x \) with the property that \( |f'(x)| < 1/2 \) for all \( x \). Define the sequence \( x_k \) by the rule \( x_1 = 1 \) and \( x_{k+1} = f(x_k) \) for \( k = 1, 2, \ldots \). Show that the \( x_k \) converge to a point \( p \) and that \( f(p) = p \), so \( p \) is a fixed point of \( f \). [SUGGESTION: Use the mean value theorem to show that \[ |x_{k+1} - x_k| \leq \frac{1}{2} |x_k - x_{k-1}| \] and then use work we did earlier to conclude that the \( x_k \) is a Cauchy sequence etc.

12. Suppose \( u \) is a twice differentiable function on \( \mathbb{R} \) which satisfies the differential equation
\[
\frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} - c(x) u = 0,
\]
where \( b(x) \) and \( c(x) \) are continuous functions on \( \mathbb{R} \) with \( c(x) > 0 \) for every \( x \in (0, 1) \).

a) Show that \( u \) cannot have a positive local maximum in the interval \((0, 1)\). Also show that \( u \) cannot have a negative local minimum in \((0, 1)\).

b) If \( u(0) = u(1) = 0 \), prove that \( u(x) = 0 \) for every \( x \in [0, 1] \).

[Last revised: November 1, 2013]