The problem numbers refer to the D’Angelo-West text.

1. [#16.1] For \( x \neq 0 \) compute \( \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{(x+h)^2} - \frac{1}{x^2} \right) \).

   **Solution:** Let \( f(x) = \frac{1}{x^2} \). Then
   \[
   \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{(x+h)^2} - \frac{1}{x^2} \right) = f'(x) = -\frac{2}{x^3}.
   \]

   Of course you can also compute the limit directly, but recognizing the limit as a derivative was my thought.

2. [#16.11] Use the definition of the derivative as the limit of a difference quotient to derive the product rule for differentiating \( f(x)g(x) \). [Suggestion: Add and subtract an appropriate quantity in the numerator.]

   **Solution:**
   
   \[
   \lim_{h \to 0} f(x+h)g(x+h) - f(x)g(x) \hfill \\
   = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \hfill \\
   = \lim_{h \to 0} f(x+h) \left( \frac{g(x+h) - g(x)}{h} \right) + \frac{f(x+h) - f(x)}{h} g(x) \hfill \\
   = \lim_{h \to 0} f(x+h) \left( \frac{g(x+h) - g(x)}{h} \right) + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} g(x) \hfill \\
   = f(x)g'(x) + f'(x)g(x).
   \]

3. Use the definition of the derivative as the limit of a difference quotient to derive the formula for the derivative of \( f(x) = \sqrt{x} \) for \( x > 0 \).

   **Solution:** For any \( x > 0 \),
   \[
   f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \hfill \\
   = \lim_{h \to 0} \left( \frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left( \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x} + h + \sqrt{x}} \right) \hfill \\
   = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
   \]
4. Let a smooth function \( g(x) \) have the properties: \( g(0) = 3 \), \( g(1) = 1 \), \( g(4) = 7 \).
   a) Show that at some point \( 0 < c < 4 \) one has \( g''(c) > 0 \). Better yet, find a number \( m > 0 \) so that \( g''(c) \geq m > 0 \).
   b) Is it true that \( g'' \) must be positive at at least one point in the interval \( 0 < x < 1 \)?
      Proof or counterexample.
   c) [This is the optimal version of part (a)]. Let \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) be any three points in the plane with \( x_1 < x_2 < x_3 \), \( y_1 > y_2 \), and \( y_3 > y_2 \). Then there is a point \( c \in (x_1, x_3) \) such that \( g''(c) = m > 0 \), where \( m \) is the second derivative of the (unique) quadratic polynomial passing through the three points.

   **Solution:** (a) By the mean value theorem, there is a \( c_1 \in (0, 1) \) such that \( g(1) - g(0) = g'(c_1)(1 - 0) \) so \( g'(c_1) = (g(1) - g(0))/(1 - 0) = -2 \). Similarly, there is a \( c_2 \in (1, 4) \) such that \( g(4) - g(1) = g'(c_2)(4 - 1) \), so \( g'(c_2) = (g(4) - g(1))/(4 - 1) = 2 \).

   Next, by the mean value theorem applied to \( g'(x) \) there is \( c \in (c_1, c_2) \) such that
   \[
   g''(c) = \frac{g'(c_2) - g'(c_1)}{c_2 - c_1} = \frac{4}{c_2 - c_1} > \frac{4}{(7 - 0)} = \frac{4}{7}.
   \]

   (b) False. Consider \( g(x) = x^3 - 4x^2 + x + 3 \), then \( g(0) = 0 \), \( g(1) = 1 \), \( g(4) = 7 \), but \( g''(x) = 6x - 8 \) which is negative for \( x \in (0, 1) \).

   (c) Let \( y = p(x) = \alpha x^2 + \beta x + \gamma \) be the quadratic polynomial passing through the three points and let \( h(x) = p(x) - g(x) \). Then \( h(0) = h(1) = h(4) = 0 \). Hence \( h'(c_1) = h'(c_2) = 0 \) for some \( 0 < c_1 < 1 < c_2 < 4 \). Thus \( h''(c) = 0 \) for some \( c \in (c_1, c_2) \), that is, \( g''(c) = p''(c) = 2\alpha > 0 \). Note that \( 2\alpha = p''(c) > 0 \) by the same argument used in part (a).

5. Let \( v(x) \) be a smooth real-valued function for \( 0 \leq x \leq 1 \). If \( v(0) = v(1) = 0 \) and \( v''(x) \geq 0 \) for all \( 0 \leq x \leq 1 \), show that \( v(x) \leq 0 \) for all \( 0 \leq x \leq 1 \).

   **Solution:** By contradiction, say \( v(x) > 0 \) for some \( x \in (0, 1) \). Then by the mean value theorem \( v'(c_1) = \frac{v(x) - v(0)}{x - 0} > 0 \) for some \( c_1 \in (0, x) \) and \( v'(c_2) = \frac{v(1) - v(x)}{1 - x} < 0 \) for some \( c_2 \in (x, 1) \). Thus, by the Mean Value theorem again, for some \( c \in (c_1, c_2) \) we have \( v''(c) = \frac{v'(c_2) - v'(c_1)}{c_2 - c_1} < 0 \). This contradicts our assumption that \( v''(x) \geq 0 \).

6. Let \( g(x) \) is a smooth function with \( g(2) = 0 \) and let \( f(x) = x^2 g(x) \). Use the mean value theorem to show that \( f''(c) = 0 \) for some \( 0 < c < 2 \).

   **Solution:** Use the Mean Value Theorem twice. We know \( f(0) = 0^2 g(0) = 0 \) and \( f(2) = 2^2 g(2) = 0 \). So \( f'(c_1) = 0 \) for some \( c_1 \in (0, 2) \). Since \( f'(x) = 2xg(x) + x^2 g'(x) \), \( f'(0) = 0 = f'(c_1) \). Hence \( f''(c) = 0 \) for some \( c \in (0, c_1) \subset (0, 2) \).

7. a) Let \( g(x) := x^3(1 - x) \). Use the mean value theorem to show that \( g''''(c) = 0 \) for some \( 0 < c < 1 \).
b) Let \( h(x) := x^3(1 - x)^3 \). Show that \( h'''(x) \) has exactly three distinct roots in the interval \( 0 < x < 1 \).

c) Let \( p(x) := \left( \frac{d}{dx} \right)^4 (1 - x^2)^4 \). Show that \( p \) is a polynomial of degree 4 and that it has 4 real distinct roots, all lying in the interval \(-1 < x < 1\).

**Solution:** (a) Since \( g(0) = g(1) = 0 \), there is \( c_1 \in (0, 1) \) such that \( g'(c_1) = 0 \). Since \( g'(0) = g'(c_1) \), there is \( c_2 \in (0, c_1) \) such that \( g''(c_2) = 0 \). Since \( g''(0) = g''(c_2) = 0 \), there is \( c \in (0, c_2) \) such that \( g'''(c) = 0 \).

(b) Since \( h(0) = h(1) = 0 \), there is \( c_1 \in (0, 1) \) such that \( h'(c_1) = 0 \). Since \( h'(0) = h'(c_1) = h'(1) = 0 \), there are \( c_2 \in (0, c_1) \) and \( c_3 \in (c_1, 1) \) such that \( h''(c_2) = h''(c_3) = 0 \). Since \( h''(0) = h''(c_2) = h''(c_3) = 0 \), there are \( c_4 \in (0, c_2) \), \( c_5 \in (c_2, c_3) \), and \( c_6 \in (c_3, 1) \) such that \( h'''(c_4) = h'''(c_5) = h'''(c_6) = 0 \). So \( h''' \) has at least 3 distinct roots and \( h''' \) has at most 3 distinct roots because it is a polynomial of degree 3.

(c) Since \( h(x) := (1 - x^2) - (1 + x)^4(1 - x)^4 \), this problem is almost identical to part (b). Note that \( p(x) = h'''(x) \). Since \( h \) is a polynomial of degree 8, \( p \) is a polynomial of degree 4. Since \( h(-1) = h(1) = 0 \), there is \( c_1 \in (-1, 1) \) such that \( h'(c_1) = 0 \).

Since \( h'(-1) = h'(c_1) = h'(1) = 0 \), there are \( c_2 \in (-1, c_1) \) and \( c_3 \in (c_1, 1) \) such that \( h''(c_2) = h''(c_3) = 0 \).

Since \( h''(-1) = h''(1) = 0 = h''(c_2) = h''(c_3) \), there are \( c_4 \in (-1, c_2) \), \( c_5 \in (c_2, c_3) \) and \( c_6 \in (c_3, 1) \) such that \( h'''(c_4) = h'''(c_5) = h'''(c_6) = 0 \).

Since \( h'''(0) = h'''(1) = 0 \), there are \( c_7 \in (-1, c_4) \), \( c_8 \in (c_4, c_5) \), \( c_9 \in (c_5, c_6) \), and \( c_{10} \in (c_6, 1) \) such that \( h'''(c_7) = h'''(c_8) = h'''(c_9) = h'''(c_{10}) = 0 \). So \( p = h''' \) has at least 4 real distinct roots and \( p \) has at most 4 real distinct roots because it is a polynomial of degree 4.

**Remark:** If in part (c) you replace the 4 by \( n \), you get the Legendre polynomial of degree \( n \). It has \( n \) real distinct zeroes in the interval \((-1, 1)\).

8. If \( b \geq 0 \), show that for every real \( c \) the equation \( x^5 + bx + c = 0 \) has exactly one real root.

**Solution:** Let \( f(x) = x^5 + bx + c \). Since \( f(a) > 0 \) and \( f(-a) < 0 \) for all sufficiently large \( a > 0 \), there is \( x \in (-a, a) \) such that \( f(x) = 0 \). Since \( b \geq 0 \), \( f \) is strictly monotone increasing, so \( f \) has at most one real root.

9. Let \( p(x) := x^3 + 3cx + d \), where \( c \), and \( d \) are real. Under what conditions on \( c \) and \( d \) does this have three distinct real roots? [Suggestion: Look at the graph of \( p \) and observe something simple about the local maximum and local minimum for \( p \) to have three distinct real roots.] [Answer: \( c < 0 \) and \( d^2 < -4c^3 \).]
Solution: Observe that \( p \) is strictly monotone increasing when \( c \geq 0 \) in which case \( p \) has exactly one real root. Thus, if \( p \) has 3 distinct real roots, then we must have \( c \leq 0 \), which we now assume. With hindsight it will be simpler if we write \( c = -\gamma \) so \( \gamma > 0 \) and \( p(x) = x^3 - 3\gamma x + d \).

Since \( p'(x) = 3x^2 - 3\gamma \), then \( p \) is strictly monotone increasing on \((-\infty, -\sqrt{\gamma})\), strictly monotone decreasing on \([-\sqrt{\gamma}, \sqrt{\gamma}]\), and strictly monotone increasing on \([\sqrt{\gamma}, \infty)\).

Because \( p''(x) = 3x \), \( p \) has a local maximum at \( x = -\sqrt{\gamma} \) and a local minimum at \( x = \sqrt{\gamma} \).

Thus \( p \) has at most one root on each of these three intervals. If \( p \) has three distinct roots, then \( p \) must has exactly one root on each of these intervals. Therefore we need \( p(-\sqrt{\gamma}) > 0 \) and \( p(\sqrt{\gamma}) < 0 \).

Conversely, by the intermediate value theorem (used thrice), if \( p(-\sqrt{\gamma}) > 0 \) and \( p(\sqrt{\gamma}) < 0 \) then \( p \) has at least (and thus exactly) three distinct roots.

We now compute \( p(\pm\sqrt{\gamma}) \):

\[
p(+\sqrt{\gamma}) = \gamma\sqrt{\gamma} - 3\gamma\sqrt{\gamma} + d = -2\gamma\sqrt{\gamma} + d.
\]

The condition \( p(+\sqrt{\gamma}) < 0 \) is thus \( d < 2\gamma\sqrt{\gamma} \).

Similarly

\[
p(-\sqrt{\gamma}) = 2\gamma\sqrt{\gamma} + d
\]

and the condition \( p(-\sqrt{\gamma}) > 0 \) is \( d > -2\gamma\sqrt{\gamma} \).

Combining them we get \(-2\gamma\sqrt{\gamma} < d < 2\gamma\sqrt{\gamma}\), that is, \( d^2 < 4\gamma^3 \).

Summarizing in terms of \( c = -\gamma \), \( p(x) = x^3 + 3cx + d \) has three real distinct roots if and only if \( c < 0 \) and \( d^2 < -4c^3 \).

Remark. The general cubic polynomial \( p(x) := x^3 + Bx^2 + Cx + D \) can be reduced to the special form here by making the substitution \( x = t - (B/3) \). You can be led to this by the observation that \( p''(x)/3! = (6x + 2B)/6 \).

10. [\#16.31] Let \( f(x) \) be a differentiable function for all real \( x \) with the property that \( f'(x) < 1 \) for all \( x \). Show has at most one fixed point, that is, at most one point \( p \) where \( f(p) = p \).

Solution: Suppose \( f \) has two distinct fixed points \( a < b \), that is, \( f(a) = a \) and \( f(b) = b \). Then by the Mean Value Theorem there is \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1 \), which contradicts the assumption that \( f'(x) < 1 \).

11. Let \( f(x) \) be a differentiable function for all real \( x \) with the property that \( |f'(x)| < 1/2 \) for all \( x \). Define the sequence \( x_k \) by the rule \( x_1 = 1 \) and \( x_{k+1} = f(x_k) \) for \( k = 1, 2, \ldots \).
Show that the $x_k$ converge to a point $p$ and that $f(p) = p$, so $p$ is a fixed point of $f$.

[Suggestion: Use the mean value theorem to show that]

$$|x_{k+1} - x_k| \leq \frac{1}{2} |x_k - x_{k-1}|$$

and then use work we did earlier to conclude that the $x_k$ is a Cauchy sequence etc.

**Solution:** For any $k \geq 2$, there is $c$ between $x_{k-1}$ and $x_k$ such that $x_{k+1} - x_k = f(x_k) - f(x_{k-1}) = f'(c)(x_k - x_{k-1})$. Since $|f'(c)| < \frac{1}{2}$, $|x_{k+1} - x_k| = |f'(c)||x_k - x_{k-1}| \leq \frac{1}{2}|x_k - x_{k-1}|$. Then $x_k$ converges because of a problem we did before concerning contracting sequences.

12. Suppose $u$ is a twice differentiable function on $\mathbb{R}$ which satisfies the differential equation

$$\frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} - c(x) u = 0,$$

where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0,1)$.

a) Show that $u$ cannot have a positive local maximum in the interval $(0,1)$. Also show that $u$ cannot have a negative local minimum in $(0,1)$.

b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0,1]$.

**Solution:** (a) Proof by contradiction. Assume that $u$ has a positive local maximum at some point $\alpha \in (0,1)$, then $u''(\alpha) \leq 0$, $b(\alpha)u'(\alpha) = 0$, and $-c(\alpha)u(\alpha) < 0$. Adding these three (in)equalities, we get $u''(\alpha) + b(\alpha)u'(-c(\alpha)u(\alpha) < 0$, but $u$ satisfies $u'' + b(x)u' - c(x)u = 0$ so we have a contradiction. Thus $u$ has no positive local maximum on $(0,1)$.

Assume that $u$ has a negative local minimum at $c \in (0,1)$, then the function $v(x) := -u(x)$ also satisfies the same equation, $v'' + bv' - cv = 0$ and would have a positive local maximum – which cannot happen by the previous paragraph. So $u$ has no negative local minimum on $(0,1)$.

(b) Since $u$ is continuous on the closed and bounded interval $[0,1]$, $u$ attains its maximum somewhere on $[0,1]$ at some $c \in [0,1]$. We have $u(c) \geq u(0) = 0$. If $u(c) > 0$, since $u(x) = 0$ at the end points of $[0,1]$, then $u$ has a positive local maximum at $c \in (0,1)$, which is impossible by part (a). Hence $u(c) = 0$, that is $u \leq 0$ on $[0,1]$. Similarly, $u \geq 0$ on $[0,1]$. Consequently $u(x) \equiv 0$ on $[0,1]$.

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