DIRECTIONS: Part A has 8 shorter problems (5 points each) while Part B has 3 traditional problems (10 points each). To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one $3 \times 5$ with notes on both sides.

PART A: Eight shorter Problems, 5 points each.

A-1. Show that $\sqrt{5}$ is not a rational number.

A-2. If $a$ and $b$ are rational numbers, consider the set $S$ of real numbers of the form $a + b\sqrt{5}$. Show that the non-zero elements in $S$ have multiplicative inverses in $S$. [This is the key step in showing that $S$ is a field.]
A-3. Determine if the set \( S = \{ x \in \mathbb{R} : 2x^2 > x^3 - 3x \} \) is bounded above and/or below, and if so, find \( \inf(S) \) and \( \sup(S) \) — if they exist.

A-4. Give an example of a sequence of real numbers that is not monotone but that does converge to some limit.

A-5. If \( x_1 \) is a given real number and \( x_{n+1} = \sqrt{1 + x_n^2} \) for \( n = 1, 2, \ldots \), show that the sequence \( x_n \) diverges.
A-6. Let $f, g : \mathbb{R} \to \mathbb{R}$ be bounded functions such that $f(x) \leq g(x)$ for all $x$. Let $F$ denote the image of $f$ and $G$ the image of $g$. Give an example (a picture) of pairs of such functions with $\sup(F) > \inf(G)$.

A-7. Compute $\lim_{n \to \infty} \frac{1 + 2n - 5n^2}{4 + 3n^2}$. Carefully note any standard theorems you use.

A-8. Give an example of a sequence $x_n$ of real numbers with at least two subsequences that converge to different limits.
PART B: Three traditional problems, 10 points each.

B-1. a) For which real numbers $c > 0$ does $\lim_{n \to \infty} n^2 c^n = 0$? Why?

b) Repeat this if $c$ is a complex number.
B-2. Let the real sequence $b_n > 0$ converge to a limit $B > 0$. Show with your bare hands (an $\epsilon$ argument) that $1/b_n \to 1/B$. [Be careful to show that any denominators are bounded away from zero.]
B-3. A sequence \( x_n \in \mathbb{R} \) is called contracting if for some constant \( 0 < c < 1 \) (such as \( c = \frac{1}{2} \)) it has the property that for all \( n = 1, 2, 3, \ldots \)

\[
|x_{n+1} - x_n| \leq c|x_n - x_{n-1}|.
\]

The point of this problem is to show that a contracting sequence converges.

a) Show that \( |x_{n+1} - x_n| \leq c^n|x_1 - x_0| \) for all \( n \).

b) Use \( x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + \cdots + (x_1 - x_0) \) to show that

\[
|x_{n+1} - x_0| \leq (c^n + c^{n-1} + \cdots + c + 1)|x_1 - x_0|
\]

c) More generally, if \( n > k \) show that

\[
|x_{n+1} - x_k| \leq \left(c^n + c^{n-1} + \cdots + c^k\right)|x_1 - x_0|
\]

\[
= c^k \left(\frac{1 - c^{n-k+1}}{1 - c}\right)|x_1 - x_0| < c^k \frac{|x_1 - x_0|}{1 - c}.
\]

Remark: Since \( 0 < c < 1 \), this shows that the \( x_n \) are a Cauchy sequence and hence converge.