Directions: Part A has 5 shorter problems (8 points each) while Part B has 4 traditional problems (15 points each). [100 points total].
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one $3 \times 5$ with notes on both sides.

Part A: Five shorter problems, 8 points each [total: 40 points]

A-1. Give an example of an infinite series $\sum a_n$ that converges but does not converge absolutely. [You do not need to justify your assertion.]
Solution: The alternating harmonic series: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$

A-2. Give an example of a bounded function defined on $-2 \leq x \leq 2$ that is continuous everywhere except at $x = 0$. [You do not need to justify your assertion].
Solution: $f(x) = \begin{cases} 0 & -2 \leq x \leq 0, \\ 1 & 0 < x \leq 2. \end{cases}$ also $f(x) = \begin{cases} \sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$

The function $g(x) = 1/x$ for $x \neq 0$ and $g(0) = 1$ is not an example. Although it is certainly not continuous at $x = 0$, it is not bounded – and the problem asks for a bounded function.

A-3. Show that the polynomial $p(x) := x^6 + x^5 - 5$ has at least two real zeroes.
Solution: For $x$ near $\pm \infty$, clearly $p(x) > 0$. Also $p(0) = -5$. Now apply the intermediate value theorem to the two intervals $x \leq 0$ and $x \geq 0$. [One could also have observed that just as obviously $p(\pm 2) > 0$ so there are two real roots in the interval $-2 < x < 2$.]

A-4. Let $g(x)$ be any smooth function and let $f(x) = (x - 1)(x - 2)(x - 3)g(x)$. Show there is a point $c \in (1, 3)$ where $f''(c) = 0$.
Solution: By Rolle’s theorem there are points $c_1$ with $1 < c_1 < 2$ and $c_2$ with $2 < c_2 < 3$ with $f'(c_1) = 0$ and $f'(c_2) = 0$. Apply Rolle’s theorem again using $f'(x)$ in the interval $[c_1, c_2]$ to obtain a point $c \in [c_1, c_2]$ with $f''(c) = 0$.

A-5. Say a function $f(x)$ has the properties $f'(x) = 2$ for all $x \in \mathbb{R}$ and $f(1) = 2$. Show that $f(x) = 2x$. [Hint: To show that “$A = B$”, it is often easiest to show that “$A - B = 0$”.
Solution: Let $g(x) = f(x) - 2 x$. Then $g'(x) = 0$ everywhere. Thus by the Mean Value Theorem $g(x) \equiv$ constant. But $g(1) = 0$ so $g(x) = 0$ everywhere.

Slight variant. Since $f'(x) \equiv 2$, by the Mean Value Theorem, there is a $c$ between 1 and $x$ so that
$$f(x) - f(1) = f'(c)(x - 1) = 2(x - 1),$$
that is $f(x) - 2 = 2(x - 1)$.
Thus, $f(x) = 2x$. 
B-1. Determine if the series \( 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \) converges or diverges. Please explain your reasoning.

Solution: 
\[ 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots > \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots = \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \right], \]
so the series diverges by comparison with the harmonic series.

B-2. Use the definition of the derivative as the limit of a difference quotient to show that if \( f(x) = \cos 2x \), then \( f \) is differentiable everywhere and compute its derivative. [You may use that \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \) and \( \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0 \).]

Solution:
\[
\frac{\cos(2(x+h)) - \cos 2x}{h} = \frac{\cos 2x \cos 2h - \sin 2x \sin 2h - \cos 2x}{h} = \frac{\cos 2x(\cos 2h - 1)}{h} - \frac{\sin 2x \sin 2h}{h} = 2 \frac{\cos 2x(\cos 2h - 1)}{2h} - 2 \frac{\sin 2x \sin 2h}{2h}.
\]
Now let \( h \to 0 \) and use \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \) and \( \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0 \) to see that
\[
\lim_{h \to 0} \frac{\cos(2(x+h)) - \cos 2x}{h} = 0 - 2 \sin 2x = -2 \sin 2x.
\]
Thus \( \cos 2x \) is differentiable everywhere and its derivative is \(-2 \sin 2x\).

B-3. Let \( f(x) \) have two continuous derivatives in the interval \((a, b)\) and say \( f''(x) \geq 0 \) for all \( x \in [a, b] \). Prove that for any \( x_0 \) the graph of \( y = f(x) \) lies above its tangent line at \((x_0, f(x_0))\). [If the equation of the tangent line at \( x_0 \) is \( y = g(x) \), then by “lies above” I mean \( f(x) \geq g(x) \) for all \( x \in [a, b] \).]

Solution: The equation of the tangent line at \( x_0 \) is \( g(x) = f(x_0) + f'(x_0)(x - x_0) \).

Method 1. Taylor’s Theorem says that there is a \( c \) between \( x_0 \) and \( x \) so that
\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(c)(x - x_0)^2.
\]
Since \( f''(c) \geq 0 \), this shows that \( f(x) \geq g(x) \) for all \( x \), that is, the curve lies above its tangent line.

Method 2. If \( x > x_0 \), then there is some \( p \in (x_0, x) \) where \( f(x) - f(x_0) = f'(p)(x - x_0) \). But because \( f''(x) \geq 0 \), we know that \( f'(p) \geq f'(x_0) \). Thus \( f(x) - f(x_0) \geq f'(x_0)(x - x_0) \), that is, \( f(x) \geq f(x_0) + f'(x_0)(x - x_0) \), as desired.

Similarly, If \( x < x_0 \), then there is some \( q \in (x, x_0) \) where \( f(x_0) - f(x) = f'(q)(x_0 - x) \). Since \( f'(q) \leq f'(x_0) \), then \( f(x_0) - f(x) \leq f'(x_0)(x_0 - x) \), that is, \( f(x) \geq f(x_0) + f'(x_0)(x - x_0) \), as desired.
B-4. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has the property that there is a constant $a > 0$ so that $f'(x) \geq a$ for all $x \in \mathbb{R}$.

a) Show that if $x \geq 0$, then $f(x) \geq f(0) + ax$ while if $x \leq 0$, then $f(x) \leq f(0) + ax$.

SOLUTION: If $x \geq 0$, by the Mean Value Theorem there is a $c_1$ in the interval $(0, x)$ where

$$f(x) - f(0) = f'(c_1)x \geq ax, \quad \text{that is,} \quad f(x) \geq f(0) + ax.$$  

Similarly, if $x \leq 0$, by the Mean Value Theorem there is a $c_2 \in (x, 0)$ where

$$f(x) - f(0) = f'(c_2)x \leq ax, \quad \text{that is,} \quad f(x) \leq f(0) + ax.$$  

b) Show that for every $c \in \mathbb{R}$ there is one (and only one) solution of the equation

$$f(x) = c.$$  

Thus, there are two steps: (i) show the equation has at least one solution and (ii) show that the equation has at most one solution.

[NOTE The existence of at least one solution may be false if you assume only $f'(x) > 0$. For example the equation $e^x = -1$ has no solution.]

SOLUTION: By part a) we see that as $x \to +\infty$ then $f(x) \to +\infty$ and as $x \to -\infty$ then $f(x) \to -\infty$ (this was the point for including Part (a)). Thus by the Intermediate Value Theorem for any $c$ there is at least one $x$ such that $f(x) = c$.

Next we show there is at most one such solution. Reasoning by contradiction, say there were two distinct solutions, $x_1 < x_2$. Then $f(x_1) = f(x_2) = c$. But by the Mean Value Theorem there is a point $\gamma$ between $c_1$ and $c_2$ where $f(x_2) - f(x_1) = f'(\gamma)(x_2 - x_1) > 0$, a contradiction. [This uniqueness part only used $f'(x) > 0$, not $f'(x) \geq a > 0$.]