DIRECTIONS: Part A has 6 shorter problems (8 points each) while Part B has 4 traditional problems (13 points each). 100 points total. To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Six shorter problems, 8 points each [total: 48 points]

A-1. Say a function \( f(x) \) has the properties \( f'(x) = 2 \cos 2x \) for all \( x \in \mathbb{R} \) and \( f(0) = 0 \). Show that \( f(x) = \sin 2x \). [HINT: To show that “\( A = B \)”, it is often easiest to show that “\( A - B = 0 \)”.]

A-2. Find the continuous function \( f \) and constant \( C \) so that \( \int_1^x f(t) \, dt = x \cos(\pi x) + C \).

A-3. Give an example of a bounded continuous function \( f(x) \), \( x \in \mathbb{R} \), that does not attain its supremum. A clear sketch is adequate.

A-4. Let \( a_n \) be a sequence of real numbers that converges to \( A \). If \( a_n \geq 0 \), give a clear proof that \( A \geq 0 \).

A-5. Give an example of a sequence, \( f_n(x) \), of functions on the interval \([0, 1]\) that converge pointwise to 0 but do not converge uniformly. A good sketch is adequate.

A-6. Let \( p(x) = x^9 + a_8x^8 + \cdots + a_1x + a_0 \). Prove (clearly) that \( \lim_{x \to -\infty} p(x) = -\infty \).

PART B: Four traditional problems, 13 points each [52 points]

B-1. Let \( f(x) \) and \( g(x) \) be differentiable for \( x \in [a, b] \) and let \( p \in (a, b) \). Show directly from the definition of the derivative that the product, \( f(x)g(x) \), is differentiable at the point \( p \) and the derivative is given by the usual rule: \( (fg)'(p) = f'(p)g(p) + f(p)g'(p) \).

B-2. Let \( f \) be a continuous function on the interval \([a, b]\). If \( \int_a^b f(x) \, dx = 0 \), show there is a point \( c \in (a, b) \) so that \( f(c) = 0 \).

B-3. Let \( I_k = \{ x \in \mathbb{R} | a_k \leq x \leq b_k \} \) be closed bounded nested intervals, so \( I_{k+1} \subseteq I_k \).

a) Use the completeness property of the real numbers (“bounded monotone sequences converge”) to show that there is at least one point in the intersection, \( \cap I_k \).
b) Give an example where the intersection is the *whole* interval $-1 \leq x \leq 1$.

B-4. Let $f(x)$ be continuous on the interval $[0, 1]$ and $g_n(x)$ be the sequence of functions in the figure. Show that

$$\lim_{n \to \infty} \int_0^1 f(x)g_n(x) \, dx = f(0).$$

**Suggestion** First do the case where $f(x) \equiv 1$. 

![Diagram](image-url)