Problem Set 10

DUE: In class Tuesday, Nov. 19. Late papers will be accepted until 1:00 PM Wednesday.

REMARC: Please re-read Chapter 16, pages 324-330 on Series of Functions and Chapter 17, pages 337-349 on Integration.

PROBLEMS

1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. If \( f'(x) \geq 0 \) everywhere, prove that \( f \) is “increasing” in the sense that if \( x < y \) then \( f(x) \leq f(y) \).

2. [#16.34] Let \( f \) be differentiable on the interval \([a, b]\), and suppose that both \( f \) and \( f' \) are positive. Prove that the function \( g := f/(1 + f) \) is bounded and increasing.

3. [#16.49] Suppose that \( f \) and \( g \) are convex (not necessarily differentiable) and \( c \in \mathbb{R} \). Which of the three functions \( f + g \), \( c \cdot f \), and \( f \cdot g \) must be convex? (Give proofs or counterexamples.)

4. Let \( f_n(x) := x^n(1 - x) \). Prove that \( f_n \to 0 \) uniformly on \([0, 1]\).

5. a) [#16.52] Which polynomials of odd degree are convex on \( \mathbb{R} \)?
   
   b) [#16.52] Characterize the polynomials of degree four that are convex on \( \mathbb{R} \) by giving a necessary and sufficient condition on the coefficients.

6. [#16.61] Let \( f_n(x) := x^2/(x^2 + n^2) \).
   a) Prove that \( f_n \) converges pointwise to 0 on \( \mathbb{R} \).
   
   b) Prove that \( f_n \) does not converge uniformly to 0 on \( \mathbb{R} \).
   
   c) Does \( f_n \) converge uniformly to 0 on the interval \([0, 100]\)? Justify your assertion.

7. Use the definition of the integral as a Riemann sum to compute \( \int_0^b x^3 \, dx \). You may use that \( \sum_{k=1}^n k^3 = \left( \frac{n(n+1)}{2} \right)^2 \).

8. Use the definition of the integral as a Riemann sum to compute \( \int_0^b \cos x \, dx \). You will need the formula for \( \cos \theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta \).

9. [#17.15] Let \( f \) be continuous on the interval \([a, b]\) and assume that \( f(x) \geq 0 \) for all \( a \leq x \leq b \). Use the definition of the integral as a Riemann sum to show that if \( \int_a^b f(x) \, dx = 0 \), then \( f(x) = 0 \) everywhere. [You will need to use that since \( f \) is continuous, if it is positive at some point, then it is positive in some interval containing the point.]