Newton’s method for finding square roots

Let \( A > 0 \) be a positive real number. We want to show that there is a real number \( x \) with \( x^2 = A \). We already know that for many real numbers, such as \( A = 2 \), there is no rational number \( x \) with this property. Formally, let \( f(x) := x^2 - A \). We want to solve the equation \( f(x) = 0 \).

Newton gave a useful general recipe for solving equations of the form \( f(x) = 0 \). Applied to compute square roots, so \( f(x) := x^2 - A \), it (see below) gives

\[
x_{k+1} = \frac{1}{2} \left( x_k + \frac{A}{x_k} \right).
\]

(1)

Clearly, if the initial approximation is positive, \( x_1 > 0 \) (we’ll assume this) then all of the \( x_k \) are positive. To get some sense of these approximations, in the special case where \( A = 3 \) and the initial approximation is \( x_1 = 1 \) I used a calculator and found (to 20 decimal accuracy)

\[
\begin{align*}
x_2 &= 2.0, \quad x_3 = 1.75, \quad x_4 = 1.7321428571428571428 \\
x_5 &= 1.7320508100147275405, \quad x_6 = 1.7320508075688772952
\end{align*}
\]

while the exact number is \( \sqrt{3} = 1.7320508075688772935 \), so \( x_6 \) above is already very close. Beginning with \( x_2 \) the successive approximations seem to be decreasing. To investigate this we compute \( x_{n+1} - x_n \). From (1), by simple algebra we find that

\[
x_{k+1} - x_k = \frac{A - x_k^2}{2x_k}.
\]

(2)

Thus, there are two cases: CASE 1 is \( x_k^2 > A \). Here \( x_{k+1} < x_k \). CASE 2 is \( x_k^2 < A \). Here \( x_{k+1} > x_k \).

If we are in Case 1 for \( x_k \), are we also in Case 1 for \( x_{k+1} \)? We compute:

\[
x_{k+1}^2 - A = \left( \frac{x_k^2 + A}{2x_k} \right)^2 - A = \frac{(x_k^2 - A)^2}{4x_k^2}.
\]

(3)

Since the right hand side is always positive (lucky!), we see that beginning with \( k = 2 \) we are always in Case 1, no matter if we start in Case 1 or Case 2. Consequently beginning with \( x_2 \) the sequence is monotone decreasing. Because it is bounded below, the \( x_k \) converge to some limit \( L > 0 \). From (2) since the left side converges to zero it is clear that \( A - L^2 = 0 \) so \( L = \sqrt{A} \).

The inequality (3) also yields a valuable estimate of the rate of convergence. This is easiest to appreciate if we look at the case where \( A \geq 1 \). Because \( x_k^2 > A > 1 \) (for \( k \geq 2 \)) we have

\[
x_{k+1}^2 - A \leq \frac{(x_k^2 - A)^2}{4A^2} \leq (x_k^2 - A)^2
\]

(4)
Thus at each step, the error, $x_{k+1}^2 - A$, is less than the square of the error in the previous step. For instance, if $x_k^2 - A < 10^{-5}$, then $x_{k+1}^2 - A < 10^{-10}$, an increase of doubling the number of decimal point accuracy. Now that we know $\sqrt{A}$ exists, it is easy to verify the related error estimate

$$x_{k+1} - \sqrt{A} = \frac{1}{2x_k}(x_k - \sqrt{A})^2.$$  

(5)

This confirms that the rapid convergence of the numerical experiment we did at the beginning was not a coincidence.

**Newton's Method** is a useful general recipe for solving equations of the form $f(x) = 0$. Say we have some approximation $x_k$ to a solution. He showed how to get a better approximation $x_{k+1}$. It works most of the time if your approximation is close enough to the solution. Here's the procedure. Go to the point $(x_k, f(x_k))$ and find the tangent line. Its equation is

$$y = f(x_k) + f'(x_k)(x - x_k).$$

The next approximation, $x_{k+1}$, is where this tangent line crosses the x axis. Thus,

$$0 = f(x_k) + f'(x_k)(x_{k+1} - x_k), \quad \text{that is,} \quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$ 

Applied to compute square roots, so $f(x) := x^2 - A$, this gives

$$x_{k+1} = \frac{1}{2} \left( x_k + \frac{A}{x_k} \right),$$

which is what we used in (1).

[Last revised: September 28, 2013]