The key to obtaining this formula is either to use some imaginative trigonometric identities or else recall that $e^{ix} = \cos x + i\sin x$ and then routinely sum a geometric series. I prefer the later. Thus

$$\sin x + \sin 2x + \cdots + \sin nx = \text{Im}\{e^{ix} + e^{i2x} + \cdots + e^{inx}\},$$

where $\text{Im}\{z\}$ means take the imaginary part of the complex number $z = x + iy$. The sum on the right side is a (finite) geometric series $t + t^2 + \cdots + t^n$ where $t = e^{ix}$:

$$t + t^2 + \cdots + t^n = \frac{t(1 - t^n)}{1 - t}.$$

Thus

$$\sin x + \sin 2x + \cdots + \sin nx = \text{Im}\left\{\frac{e^{ix}(1 - e^{inx})}{1 - e^{ix}}\right\}.$$  

(2)

We need to find the imaginary part of the fraction on the right. The denominator is what needs work. By adding and subtracting

$$e^{i\theta} = \cos \theta + i\sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i\sin \theta$$

we obtain the important formulas

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$ 

Thus

$$1 - e^{ix} = e^{ix/2}\left(e^{-ix/2} - e^{ix/2}\right) = -2ie^{ix/2}\sin \frac{x}{2}$$

so

$$\frac{e^{ix}(1 - e^{inx})}{1 - e^{ix}} = i\left[\frac{e^{ix/2} - e^{i(n+1/2)x}}{2\sin \frac{x}{2}}\right].$$  

(3)

Consequently, from (3), taking the imaginary part of the right side (so the real part of $\cdots$) we obtain the desired formula:

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\cos \frac{1}{2}x - \cos (n + \frac{1}{2})x}{2\sin \frac{1}{2}x}$$

REMARK: By taking the real part in (3) we obtain the related formula

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{-\sin \frac{1}{2}x + \sin (n + \frac{1}{2})x}{2\sin \frac{1}{2}x}.$$  

Exercise: Use $\sin(a + x) + \sin(a + 2x) + \cdots + \sin(a + nx) = \text{Im}\{e^{ia}(e^{ix} + \cdots + e^{inx})\}$ to compute a formula for $\sin(a + x) + \sin(a + 2x) + \cdots + \sin(a + nx)$. [Taking the derivative of this formula with respect to $a$ gives another route to the formula of the Remark just above.]

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