Symmetries of a Square

EXAMPLE To describe the symmetries of a square $ABCD$, introduce coordinates so that the center of the square is at the origin. One obvious symmetry is a 90 degree counterclockwise rotation $R$. Then $R^2$ (just repeat $R$) is the rotations by 180 degrees. Also $R^3$ is the rotation by 270 degrees – which is clearly equivalent to a clockwise rotation by 90 degrees, which we write as $R^{-1} = R^3$. A rotation by 360 degrees is the same as no rotation, so $R^4$ is the identity matrix: $R^4 = I$. Observe $R^{-1}R = R^3R = R^4 = I$, as one should want.

Another evident symmetry is the reflection, $S$, across the vertical line $PQ$. Clearly reflecting twice brings you back home, so $S^2 = I$.

We can use a sequence of these symmetries, such as $SR$ (a rotation $R$ followed by a reflection $S$), to get the complete group of symmetries of the square. The complete list of elements of this group are:

$$I, \ R, \ R^2, \ R^3, \ S, \ SR, \ SR^2, \ SR^3. \ \ (1)$$

Note that by a computation, $S^2 = I, RS = SR^3, R^2S = SR^2$, and $R^3S = SR$ so the above list contains all possible combinations of products of $R$’s and $S$’s. Since $SR \neq RS$, this group of symmetries is not commutative.

There are some additional evident symmetries of the square, for example the reflection $T$ across the horizontal line $MN$. Is this missing from our list (1)? If you sketch the figures, you will see that you can achieve $T$ by first using the reflection $S$ followed by $R^2$. Thus, $T = R^2S$. Similarly, the reflection across the diagonal $DB$ is equivalent to $RS$. The list (1) really does contain all the symmetries of the square.

EXERCISE:

a) Use $RS = SR^3$ to show that the maps $RSR, R^2S$, and $RSR^{-1}$ are in the list (1).

b) Prove that the list (1) really does contain all the symmetries of the square. I suggest beginning with the special case where the vertex $A$
is fixed. What are the possible adjacent vertices? A key ingredient is that the symmetries of the square are *rigid motions*, that is, they preserve distances between points, so no stretching or shrinking is allowed.