DIRECTIONS: Part A has 8 shorter problems (5 points each) while Part B has 4 traditional problems (10 points each). To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Eight shorter problems, 5 points each.

A-1. If \(a_1 = 1\) and \(a_{n+1} = \sqrt{3a_n + 4}\), show that \(a_n < 4\) for all \(n = 1, 2, 3, \ldots\).

SOLUTION: Use induction.

A-2. Show that \(\sqrt{3}\) is not a rational number.

SOLUTION: Routine

A-3. Show that \(\lim_{n \to \infty} \frac{5^n}{n!} = 0\).

SOLUTION: Observe that \(a_{11} = \frac{5^{11}}{11!} = \frac{5^{10}}{10!} \cdot \frac{5}{11} < \frac{a_{10}}{2}\)

Similarly, \(a_{12} < \frac{a_{11}}{2} < \frac{a_{10}}{2^2}\) Each of the subsequent terms is less than \(1/2\) its predecessor, that is, for \(n \geq 10: 0 < a_{n+1} < a_n/2\). Thus the sequence converges to 0.

OR: use the ratio test for sequences: \(|a_{n+1}/a_n| = 5/(n+1) \to 0 < 1\).

A-4. Give an example of a sequence of real numbers that is not monotone but that converges to some limit.

SOLUTION: Two examples: \(a_n = (-1)^n/n \to 0, \ b_n = 5^n/n! \to 0\).

A-5. Give an example of a sequence \(x_n\) of real numbers with at least two subsequences that converge to different limits.

SOLUTION: Two examples: \(a_n = (-1)^n, \ b_n = (-1)^n(2 + \frac{1}{n})\)

A-6. Give an example of an unbounded sequence of real numbers \(a_n\) that satisfies \(|a_{n+1} - a_n| \to 0\).

SOLUTION: Three examples: \(a_n = \sqrt{n}, \ b_n = \ln n, \ c_n = 1 + 1/2 + 1/3 + \cdots + 1/n\).

A-7. Determine if the series \(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} + \cdots\) converges or diverges. Explain your reasoning.

SOLUTION:

\[
1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n+2} = \frac{1}{2} \left[ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1} \right] \to \infty
\]
Equivalently:
\[
\sum_{k=0}^{n} \frac{1}{2k+1} > \sum_{k=0}^{n} \frac{1}{2k+2} = \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k+1}
\]

which diverges to infinity.

A-8. [Proof or Counterexample] Let \(a_n\) be a sequence of real numbers that converges to \(L\). Then there exists an \(\epsilon > 0\) such that for all integers \(n\) we have \(|a_{n+1} - a_n| < \epsilon\).

**Solution:** The key is that the inequality \(|a_{n+1} - a_n| < \epsilon\) needs to hold for all \(n\), not just for \(n\) sufficiently large. Since convergence concerns only terms with \(n\) sufficiently large, we don’t have much control over the terms at the “beginning” of the sequence. But since the sequence converges, we at least know that it is bounded: for some \(M\) we have \(|a_n| < M\) for all \(n\).

At this stage, for me it is simplest to rename \(\epsilon\) to, say, \(Q\). Thus:

“Then there exists a \(Q > 0\) such that for all integers \(n\) we have \(|a_{n+1} - a_n| < Q\).”

This assertion is correct. We see that all that is required is that the sequence be bounded, say \(|a_n| < M\). Then \(|a_{n+1} - a_n| < |a_{n+1}| + |a_n| \leq 2M\) so we can pick \(Q = 2M\).

Part B: Four traditional problems, 10 points each.

B-1. Let \(a_n\) and \(b_n\) be sequences of of complex numbers. If \(a_n \to A\) and \(b_n \to B\), show that \(a_nb_n \to AB\). [Give a formal proof using \(\epsilon\) and \(N\).]

**Solution:** Standard (and essential) preliminary:

\[|a_nb_n - AB| = |(a_n - A)b_n + A(b_n - B)| \leq |a_n - A||b_n| + |A||b_n - B|\]

Now one needs the key ingredient that since the sequence \(b_n\) converges, it is bounded: \(|b_n| < M\) for some \(M\).

The rest is routine: Given \(\epsilon > 0\), pick \(N\) so that if \(n > N\) then \(|a_n - A|< \epsilon/2\) and \(|A||b_n - B| < \epsilon/2\). Then, by the above:

\[|a_nb_n - AB| < \epsilon/2 + \epsilon/2 = \epsilon.\]

B-2. Let \(a_n = \sqrt{n^2 + 6n} - n\). Show that \(a_n\) converges and find the limit.

**Solution:** A standard device:

\[
a_n = \left(\sqrt{n^2 + 6n} - n\right)\left(\frac{\sqrt{n^2 + 6n} + n}{\sqrt{n^2 + 6n} + n}\right) = \frac{n^2 + 6n - n^2}{\sqrt{n^2 + 6n} + n} = \frac{6n}{\sqrt{n^2 + 6n} + n} \to 3.
\]
B-3. Let \(a_n\) be a sequence of real numbers that converge to \(L\). If \(L > 0\), show there is an \(N\) so that if \(n > N\) then \(a_n > \frac{1}{2}L\).

**Solution:** Pick \(N\) so that if \(n > N\) then \(|a_n - L| < \frac{1}{2}L\). In particular, \(-\frac{1}{2}L < a_n - L\), that is, \(\frac{1}{2}L < a_n\).

B-4. Let \(a_n \to A\) and \(b_n \to B\) be convergent sequences of real numbers, and let \(c_n\) be the larger of \(a_n\) and \(b_n\), so \(c_n = \max(a_n, b_n)\). Either prove that this sequence \(c_n\) converges or give a counterexample.

**Remark:** There are three cases: \(A < B\), \(A > B\), and \(A = B\).

**Solution:** If \(A < B\), then for all sufficiently large \(n\) we have \(a_n < b_n\) so \(c_n = b_n \to B\). The case \(A > B\) is essentially identical.

If \(A = B\), given \(\epsilon > 0\) pick \(N\) so that if \(n > N\) then both \(|a_n - A| < \epsilon\) and \(|b_n - A| < \epsilon\). Since either \(c_n = a_n\) or \(c_n = b_n\) then \(|c_n - A| < \epsilon\).