Part A: Eight shorter problems, 5 points each.

A-1. Find all points in the complex plane where \( \sum_{n=0}^{\infty} \frac{n}{(z-2)^n} \) converges.

**Solution:** By the ratio test, this converges absolutely when \( \left| \frac{1}{z-2} \right| < 1 \), that is, when \( |z-2| > 1 \). This is the exterior of a disk centered at \( z = 2 \) with radius 1.

The series diverges at every point of the boundary of this disk since at these points \( n / |z-2| \) equals \( n \). This uses: “If a series \( \sum c_n \) converges then \( |c_n| \to 0 \).

A-2. This problem concerns the continuity of \( f(x) = \frac{1}{x} \) at the point \( a = 1/1000 \). Let \( \epsilon = 1 \).

Find a \( \delta > 0 \) so that if \( |x-a| < \delta \) then \( |f(x) - f(a)| < \epsilon \).

**Solution:** We want a \( \delta \) so that if \( |x - \frac{1}{1000}| < \delta \), then \( |\frac{1}{x} - 1000| < 1 \), that is, \( 999 < \frac{1}{x} < 1001 \); equivalently, \( \frac{1}{1001} < x < \frac{1}{999} \). This means

\[
\frac{1}{1001} - \frac{1}{1000} < x - a < \frac{1}{999} - \frac{1}{1001},
\]

which is satisfied if \( \delta < \frac{1}{1000} - \frac{1}{1001} = \frac{1}{1001,000} \). To be less exact, we can let \( \delta = 10^{-7} \).

A-3. Give an example of a bounded continuous function \( f(x) \), \( x \in \mathbb{R} \), that does not attain its infimum. A clear sketch is adequate.

**Solution:** \( \frac{1}{1+x^2} \).

A-4. Say a function \( f(x) \) has the properties \( f'(x) = 2 \cos 2x \) for all \( x \in \mathbb{R} \) and \( f(0) = 0 \). Show that \( f(x) = \sin 2x \). [HINT: To show that \( “A = B”, \) it is often easiest to let \( C = A - B \) and then show that \( “C = 0” \).]

**Solution:** Let \( h(x) := f(x) - \sin 2x \).

I show that \( h(x) = 0 \). First, \( h'(x) = 0 \) so \( h(x) = \) constant. But \( h(0) = 0 \).

A-5. Let \( f(x) \) and \( g(x) \) be continuous on \( [a,b] \). If \( f(a) > g(a) \) and \( f(b) < g(b) \), prove that there is some \( c \in (a,b) \) where \( f(c) = g(c) \).
Solution Let \( h(x) := f(x) - g(x) \) and note that \( h(a) > 0 \) while \( h(b) < 0 \). Now apply the intermediate value theorem.

Can there be more than one such point?

Solution Yes, lots. Look at the graphs of \( f(x) = \cos x \) and \( g(x) = \sin x \) for \( 0 \leq x \leq 3\pi \)

A-6. Give an example of a function \( f(x) \) that is continuous at every point of the set \( \{ x \geq 1 \} \) but is not uniformly continuous in this set.

Solution \( x^2 \)

A-7. Give an example of a function \( f(x) \) that is continuous for \(-1 \leq x \leq 1\) but not differentiable at, say, \( x = 0 \).

Solution \( |x| \)

A-8. Let \( f(x) \), \( g(x) \), and \( h(x) \) be smooth functions

a) If \( f(a) = 0 \) and \( f'(x) \geq 0 \) for all \( x \geq a \), show that \( f(x) \geq f(a) \) for all \( x \geq a \).

Solution By the Mean Value Theorem there is a point \( c \), \( a < c < x \) so that
\[
 f(x) - f(a) = f'(c)(x - a)
\]

Since \( f'(c) \geq 0 \), then \( f(x) - f(a) \geq 0 \).

b) If \( g(a) = h(a) \) and \( g'(x) \geq h'(x) \) for all \( x \geq a \), show that \( g(x) \geq h(x) \) for all \( x \geq a \).

Solution Let \( f(x) = g(x) - h(x) \) and apply part a).

Part B: Four traditional problems, 10 points each.

B-1. Use the definition of the derivative as the limit of a difference quotient to show that if \( f(x) = \cos 2x \), then \( f \) is differentiable everywhere and compute its derivative. [You may use that \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \) and \( \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0 \).]

B-2. Let \( f(x) \) be a smooth function with the properties \( f(0) = 3 \), \( f(1) = 1 \), and \( f(3) = 5 \). Show that \( f''(c) \geq A > 0 \) for some \( c \in (0, 3) \) and some \( A > 0 \). Give an explicit value for the constant \( A \).

Solution: By the Mean Value Theorem for the intervals \( 0 \leq x \leq 1 \) and \( 1 \leq x \leq 3 \) there are points \( c_1 \in (0, 1) \) and \( c_2 \in (1, 3) \) so that
\[
 f'(c_1) = \frac{f(1) - f(0)}{1 - 0} = -2 \quad f'(c_2) = \frac{f(3) - f(1)}{3 - 1} = 2.
\]

Now apply the Mean Value Theorem again to \( f'(x) \) for \( c_1 < x < c_2 \) to find a \( c_3 \in (c_1, c_2) \) do that
\[
 f''(c_3) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{4}{c_2 - c_1} \geq \frac{4}{3}.
\]
B-3. Let $f(x)$ be differentiable at every point of the open interval $a < x < b$ (possibly unbounded).

a) If the derivative is bounded, say $|f'(x)| \leq M$, in this interval, show that $f$ is uniformly continuous in the interval.

b) If the derivative is not bounded in this interval, show that $f$ is not uniformly continuous in the interval.

**SOLUTION:** This assertion is FALSE. All of the counterexamples below are uniformly continuous – although their first derivatives are unbounded:

- $f(x) = \sqrt{x}$ for $0 \leq x \leq 1$ (the simplest example).
- $g(x) = x \sin(1/x)$ for $0 < x \leq 1$, $g(0) = 0$,
- $h(x) = \frac{\sin x^3}{x}$ for $1 \leq x$.

c) Apply these to the functions $x^2$ and $1/x$ on the interval $x \geq 1$.

B-4. a) Say the smooth function $w(x)$ satisfies $w'' - c(x)w \leq 0$, where $c(x) > 0$. Show there is no point $p$ where $w$ has a local minimum and $w(p) < 0$.

**SOLUTION:** At a local minimum $w'' \geq 0$. Since $w(p) < 0$ and $c(x) > 0$, this contradicts $w'' - c(x)w \leq 0$.

b) If on a bounded interval $a \leq x \leq b$ $w$ satisfies this and $w(a) = w(b) = 0$, show that $w(x) \geq 0$ on the whole interval.

**SOLUTION:** Reasoning by contradiction, say $w(p) < 0$ somewhere in $[a, b]$. Let $q$ be the point in $[a, b]$ where $w$ has its minimum value. Then $w(q) < 0$. Note, $q$ can’t be an endpoint because $w(a) = w(b) = 0$. Therefore $w$ has a negative local minimum at $q$. By part a), this is impossible. Therefore $w(x) \geq 0$ for all $x \in [a, b]$.

c) Say on the interval $[a, b]$ the smooth functions $u(x)$ and $v(x)$ satisfy

$$u'' - c(x)u = f(x), \quad v'' - c(x)v = g(x), \quad \text{with} \quad u(a) = v(a), \quad u(b) = v(b),$$

where, as above, $c(x) > 0$, and $f$ and $g$ are given functions. If $f(x) \leq g(x)$, show that $u(x) \geq v(x)$ in $[a, b]$.

**SOLUTION:** Let $w(x) := u(x) - v(x)$. Then by part b) we have $w(x) \geq 0$.

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1This whole problem – with the same proof – is valid for the more general differential operator $w'' + b(x)w' - c(x)w$, where $b(x)$ is any continuous function.