Directions: Part A has 6 short questions (5 points each), Part B has 2 shorter problems (8 points each), Part C has 4 traditional problems (12 points each). 94 points total. To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 × 5 with notes on both sides.

Part A: Six shorter problems, 5 points each [total: 30 points]

A-1. Give an example of a power series \( \sum_{k=0}^{\infty} a_k x^k \) that converges for all \( x \) with \( |x| < 2 \) but not if \( |x| \geq 2 \).

A-2. Let \( p(x) = x^3 - 3x + 1 \). Show that \( p(x) \) has 3 distinct real zeros.

A-3. Give an example of a sequence, \( f_n(x) \), of bounded functions on the interval \([0, 1]\) that converge pointwise but do not converge uniformly. A good sketch is adequate.

A-4. Find a continuous function \( f \) and a constant \( C \) so that \( \int_0^x f(t)(1 + t^2) \, dt = x + \cos x + C \).

A-5. Show that the series \( \sum_{k=0}^{\infty} \frac{1 + \cos 2kx}{1 + k^4} \) converges uniformly.

A-6. Say a function \( f(x) \) has the properties \( f'(x) = \frac{2x}{1 + x^2} \) for all \( x \in \mathbb{R} \) and \( f(0) = -1 \). Show that \( f(x) = \ln(1 + x^2) - 1 \).

Part B: Two shorter problems, 8 points each [16 points]

B-1. Show that \( f(x) = \frac{1}{x} \) is uniformly continuous in the set \( \{ x \geq 1 \} \).

B-2. Let \( a_n \) and \( b_n \) be sequences with the properties \( a_n \to L \) and \( b_n - a_n \to 0 \). Given any \( \epsilon > 0 \), show that \( b_n \to L \) by finding an \( N \) so that if \( n > N \) then \( |b_n - L| < \epsilon \).

Part C: Four traditional problems, 12 points each [48 points]

C-1. Let \( f(x) \) be a continuous function on the interval \( I = \{ a \leq x \leq b \} \) and let \( \mathcal{P} \) be a partition of \( I \) into two intervals having equal width \( h = (b - a)/2 \). If \( f \) is an increasing function, Show that the upper and lower Riemann sums satisfy

\[
U(f, \mathcal{P}) - L(f, \mathcal{P}) = [f(b) - f(a)]h.
\]

[Your solution should include a sketch.]
C-2. a) Let \( f(x) \) have two continuous derivatives on \( \mathbb{R} \) and let \( x_0 < x_1 < x_2 \) be given points. If 
\( f(x_0) = f(x_1) = f(x_2) = 0 \), show that there is a point \( c \in (x_0, x_2) \) where \( f''(c) = 0 \).

b) Let \( h(x) \) have two continuous derivatives on \( \mathbb{R} \) and let \( p(x) = Ax^2 + Bx + C \). If 
\[ h(x_0) = p(x_0), \quad h(x_1) = p(x_1), \quad \text{and} \quad h(x_2) = p(x_2), \]
show there is a point \( c \in (x_0, x_2) \) where \( h''(c) = p''(c) = 2A \).

C-3. If \( f \) is a continuous function on the interval \([a, b]\), let \( m := \min_{x \in [a, b]} f(x) \) and \( M := \max_{x \in [a, b]} f(x) \).

a) Show that 
\[ m \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq M. \]

b) Show there is a point \( c \in [a, b] \) such that 
\[ \frac{1}{b-a} \int_{a}^{b} f(x) \, dx = f(c) \]

C-4. Let \( f(x) \) be continuous on the interval \([0, 1]\). Show that 
\[ \lim_{n \to \infty} n \int_{0}^{1} f(x)x^n \, dx = f(1). \]