PART A: Six shorter problems, 5 points each [total: 30 points]

A-1. Give an example of a power series \( \sum_{k=0}^{\infty} a_k x^k \) that converges for all \( x \) with \( |x| < 2 \) but not if \( |x| \geq 2 \).

**SOLUTION:** The geometric series \( \sum_{k=0}^{\infty} \frac{x^k}{2^k} \)

A-2. Let \( p(x) = x^3 - 3x + 1 \). Show that \( p(x) \) has 3 distinct real zeros.

**SOLUTION:** Observe that \( p(-\infty) = -\infty \), \( p(0) = 1 \), \( p(1) = -1 \), and \( p(+\infty) = +\infty \). Now apply the intermediate value theorem.

One could also exploit that \( p \) has critical points at \( x = \pm 1 \).

A-3. Give an example of a sequence, \( f_n(x) \), of bounded functions on the interval \([0, 1]\) that converge pointwise but do not converge uniformly. A good sketch is adequate.

**SOLUTION:** \( f_n(x) = x^n \).

A-4. Find a continuous function \( f \) and a constant \( C \) so that

\[
\int_0^x f(t)(1 + t^2) \, dt = x + \cos x + C.
\]

**SOLUTION:** To find \( C \) let \( x = 0 \): \( 0 = 0 + 1 + C \) so \( C = -1 \).

To find \( f \) use the fundamental theorem of calculus:

\[
f(x)(1 + x^2) = 1 - \sin x \quad \text{so} \quad f(x) = \frac{1 - \sin x}{1 + x^2}.
\]

A-5. Show that the series \( \sum_{k=0}^{\infty} \frac{1 + \cos 2^k x}{1 + k^4} \) converges uniformly.

**SOLUTION:** Since \( \left| \frac{1 + \cos 2^k x}{1 + k^4} \right| \leq \frac{2}{1 + k^4} \), this is a consequence of the Weierstrass M Test.
A-6. Say a function \( f(x) \) has the properties \( f'(x) = \frac{2x}{1+x^2} \) for all \( x \in \mathbb{R} \) and \( f(0) = -1 \). Show that \( f(x) = \ln(1 + x^2) - 1 \).

**Solution:** Let \( g(x) = f(x) - [\ln(1 + x^2) - 1] \). Then \( g'(x) = 0 \) so \( g(x) = \text{constant} \). But \( g(0) = 0 \).

**Part B:** Two shorter problems, 8 points each [16 points]

B-1. Show that \( f(x) = 1/x \) is uniformly continuous in the set \( \{ x \geq 1 \} \).

**Solution:** Version 1. For all \( x, y \geq 1 \) we have

\[
\left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{x - y}{xy} \right| \leq |x - y|
\]

so given \( \epsilon > 0 \) pick \( \delta = \epsilon \).

Version 2. Because \( x, y \geq 1 \), then \( |f'(x)| = 1/x^2 \leq 1 \). Then by the Mean Value Theorem

\[
|f(x) - f(y)| \leq |x - y|
\]

so we can let \( \delta = \epsilon \).

B-2. Let \( a_n \) and \( b_n \) be sequences with the properties \( a_n \to L \) and \( b_n - a_n \to 0 \). Given any \( \epsilon > 0 \), show that \( b_n \to L \) by finding an \( N \) so that if \( n > N \) then \( |b_n - L| < \epsilon \).

**Solution:** Given \( \epsilon > 0 \).

There is an \( N_1 \) so that if \( n > N_1 \) then \( |a_n - L| < \epsilon/2 \).

There is an \( N_2 \) so that if \( n > N_2 \) then \( |b_n - a_n| < \epsilon/2 \).

Let \( N = \max\{N_1, N_2\} \). Then for \( n > N \)

\[
|b_n - L| = |b_n - a_n + a_n - L| \leq |b_n - a_n| + |a_n - L| \leq \epsilon/2 + \epsilon/2 = \epsilon
\]

**Part C:** Four traditional problems, 12 points each [48 points]

C-1. Let \( f(x) \) be a continuous function on the interval \( I = \{ a \leq x \leq b \} \). and let \( \mathcal{P} \) be a partition of \( I \) into two intervals having equal width \( h = (b - a)/2 \). If \( f \) is an increasing function, Show that the upper and lower Riemann sums satisfy

\[
U(f, \mathcal{P}) - L(f, \mathcal{P}) = [f(b) - f(a)]h.
\]

[Your solution should include a sketch.]

**Solution:**

\[
U(f, \mathcal{P}) = + f(a + h)h + f(b)h
\]

\[
L(f, \mathcal{P}) = f(a)h + f(a + h)h
\]

Thus

\[
U(f, \mathcal{P}) - L(f, \mathcal{P}) = [f(b) - f(a)]h
\]

2
C-2. a) Let $f(x)$ have two continuous derivatives on $\mathbb{R}$ and let $x_0 < x_1 < x_2$ be given points. If $f(x_0) = f(x_1) = f(x_2) = 0$, show that there is a point $c \in (x_0, x_2)$ where $f''(c) = 0$.

**Solution:** By Rolle’s Theorem there is a point $c_1 \in (x_0, x_1)$ so that $f'(c_1) = 0$. Similarly, there is a point $c_2 \in (x_1, x_2)$ so that $f'(c_2) = 0$. Thus there is a point $c \in (c_1, c_2)$ so that $f''(c) = 0$.

b) Let $h(x)$ have two continuous derivatives on $\mathbb{R}$ and let $p(x) = Ax^2 + Bx + C$. If $h(x_0) = p(x_0)$, $h(x_1) = p(x_1)$, and $h(x_2) = p(x_2)$, show there is a point $c \in (x_0, x_2)$ where $h''(c) = p''(c) = 2A$.

**Solution:** Apply part a) to $f(x) := h(x) - p(x)$.

C-3. If $f$ is a continuous function on the interval $[a, b]$, let $m := \min_{x \in [a,b]} f(x)$ and $M := \max_{x \in [a,b]} f(x)$.

a) Show that

$$m \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq M.$$  

**Solution:** This is obvious from the Riemann sum definition of the integral.

b) Show there is a point $c \in [a, b]$ such that

$$\frac{1}{b-a} \int_a^b f(x) \, dx = f(c).$$

**Remark:** A useful routine generalization is: for any continuous $w(x) \geq 0$

$$\int_a^b f(x)w(x) \, dx = f(c) \int_a^b w(x) \, dx$$

**Solution** Let $Q := \frac{1}{b-a} \int_a^b f(x) \, dx$. By Part a), $m \leq Q \leq M$. Thus by the Intermediate Value Theorem there is some $c \in [a, b]$ so that $f(c) = Q$.

C-4. Let $f(x)$ be continuous on the interval $[0, 1]$. Show that

$$\lim_{n \to \infty} n \int_0^1 f(x)x^n \, dx = f(1).$$

**Solution:** [This problem is more difficult than I intended.] Write $J_n(f) = n \int_0^1 f(x)x^n \, dx$.

**Method 1** To start, note that by a short computation the assertion is true for the special case where $f(x) = \text{constant}$. Write $f(x) = [f(x) - f(1)] + f(1)$, so $J_n(f) = J_n(f(x) - f(1)) + J_n(f(1))$. Since the assertion is true for the constant function $f(1)$, we need only prove it for the function $g(x) := f(x) - f(1)$ which has the additional property that $g(1) = 0$. 

3
Examining the integrand more closely, note that if $0 \leq x \leq c < 1$ then $nx^n \leq nc^n \to 0$ as $n \to \infty$. Thus, all of the action takes place near $x = 1$. This leads us to write

$$J_n(g) = \int_0^1 g(x)nx^n \, dx = \int_0^c + \int_c^1 = I_1 + I_2$$

To show that $J_n \to 0$, we will first choose $c$ near $1$ so that $|I_2| < \epsilon/2$ for all $n$. We will then pick $N$ so that if $n > N$ then $|I_1| < \epsilon/2$. Assuming for the moment that we have done this, then

$$|J_n| \leq |I_1| + |I_2| < \epsilon,$$

as desired.

To estimate $I_2$, since $g(1) = 0$ we can pick $\delta$ so that if $|x - 1| < \delta$ then $|g(x)| < \epsilon/2$ and let $c = 1 - \delta$. With this choice

$$|I_2| \leq \int_0^1 |g(x)|nx^n \, dx \leq (\epsilon/2)\int_0^1 nx^n \, dx < \epsilon/2.$$

To estimate $I_1$, let $M = \max_{[0,1]} |f(x)|$. Then because $0 \leq c < 1$, for $n$ large we have

$$|I_1| \leq M \int_c^0 nx^n \, dx = \frac{n}{n+1} M c^{n+1} < \epsilon/2.$$  

**Method 2.** For the moment we will assume that $f(x)$ is smooth ($f \in C^1([0,1])$ is enough) and integrate by parts:

$$J_n(f) = \int_0^1 f(x)x^n \, dx = \frac{n}{n+1} f(x)x^{n+1} \bigg|_0^1 - \frac{n}{n+1} \int_0^1 f'(x)x^{n+1} \, dx = \frac{n}{n+1} f(1) - \frac{n}{n+1} \int_0^1 f'(x)x^{n+1} \, dx.$$  

To estimate the second term, say $K := \max_{[a,b]} |f'(x)|$. Then

$$\left| \frac{n}{n+1} \int_0^1 f'(x)x^{n+1} \, dx \right| \leq K \frac{n}{(n+1)(n+2)}$$

Now let $n \to \infty$ in equation (??).

One can apply this even if $f$ is only continuous. Use the fact that there is a smooth function $g$ (even a polynomial) that approximates $f$ uniformly on $[0,1]$:

$$\max_{[a,b]} |f(x) - g(x)| < \epsilon/3$$

The rest is routine:

$$J_n(f) - f(1) = J_n(f-g) + [J_n(g) - g(1)] + [g(1) - f(1)].$$

But

$$|J_n(f-g)| \leq J_n(|f-g|) < (\epsilon/3)J_n(1) < \epsilon/3,$$

while, since $g$ is differentiable we know that for $n$ large, $|J_n(g) - g(1)| < \epsilon/3$. Also, $|f(1) - g(1)| < \epsilon/3$.

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Weierstrass Approximation Theorem

1