Problem Set 10
DUE: In class Thursday, Nov. 29 Late papers will be accepted until 1:00 PM Friday.

PROBLEMS

1. [# 17.6] Let $f$ and $g$ be bounded real-valued functions on a set $S$.
   a) Prove that $\sup_S (f + g) \leq \sup_S f + \sup_S g$.
   b) Give an example where strict inequality holds.

2. [#17.7] Let $f(x) = x^2$, and let $P_n$ be a partition of the interval $[0, 3]$ into $n$ intervals of equal length.
   a) Compute formulas for $L(f, P_n)$ and $U(f, P_n)$ in terms of $n$. Verify that they have the same limit.
   b) Find a number $n$ to insure that $U(f, P_n)$ is within .01 of $\int_0^3 x^2 \, dx$.

3. Let $f(x) = 0$ for all $x \in [0, 2]$ except at $x = 1$ where $f(1) = 3$. Show that $f$ is Riemann integrable on $[0, 2]$ – and compute the value of the integral.

4. Let $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = -3$. Show that $f$ is Riemann integrable on the interval $[-1, 1]$.

5. [#17.13] Let $f(x)$ be continuous for $x \in [a, b]$.
   a) Show there is some point $c \in [a, b]$ where $f$ has its average value, that is,
      \[ f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx \]
      [SUGGESTION: First do the case where $\int_a^b f(x) \, dx = 0$.
   b) If $f$ is not continuous, there may not be any such point $c$. Give an example.

6. If $\int_0^x f(t) \, dt = x \cos(\sin x) + C$, find the continuous function $f$ and the constant $C$.

7. [#17.16] For $x > 0$ let $g(x) := \int_0^x \frac{1}{1+t^2} \, dt + \int_0^{1/x} \frac{1}{1+t^2} \, dt$. Show that $g$ is a constant.
8. For which powers $p > 0$ does the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converge? [HINT: integral test.]

9. Compute $\lim_{\lambda \to \infty} \int_{0}^{1} |\sin(\lambda x)| \, dx$.

10. Let $f(t)$ be a continuous function for $0 \leq t < \infty$. If $\lim_{t \to \infty} f(t) = c$, show that

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(t) \, dt = c.$$  

[HINT: This is like Proposition 14.11 on page 275 in the text.]

**Bonus Problems**

[Please give your solutions directly to Professor Kazdan]

1-B Let $f$ be continuous on the interval $[0, \pi]$. Show that $\lim_{\lambda \to \infty} \int_{0}^{\pi} f(x) \sin(\lambda x) \, dx = 0$.  

[Last revised: November 26, 2018]