Problem Set 6

Due: In class Thursday, Oct. 18. Late papers will be accepted until 1:00 PM Friday.

Remarks:
Please re-read Chapter 15 on Continuity and read Chapter 16, pages 307-317 on Differentiation.

Problems

1. In class, if $z = x + iy$ we defined $e^z$ by a power series and observed that, $e^{ix} = \cos x + i \sin x$. From the power series one can also show that for any complex $z$ and $w$ the usual formula $e^{z+w} = e^z e^w$ remains valid. Use the observation that

$$1 + \cos x + \cos 2x + \cdots + \cos nx = \text{Real part of } \{1 + e^{ix} + e^{2ix} + \cdots + e^{nix}\}$$

and that the right hand side is a geometric series to find a formula for $1 + \cos x + \cos 2x + \cdots + \cos nx$. [Assume $x$ is not a multiple of $2\pi$]. Your resulting formula should not have any complex numbers.

2. [#15.3] [T/F] If $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and $f(x) = 0$ for all rational numbers $x$, then $f(x) = 0$ for all real $x$.

3. [#15.5] [T/F] The function $f(x) := |x|^3$ is continuous for all $x \in \mathbb{R}$.

4. [#15.7] [T/F] Let $f$, $g$, and $h$ be continuous on the interval $[0, 2]$. If $f(0) < g(0) < h(0)$ and $f(2) > g(2) > h(2)$, then there exists some $c \in [0, 2]$ such that $f(c) = g(c) = h(c)$.

5. [#15.10][T/F]
   a) If $f$ is continuous on $\mathbb{R}$, then $f$ is bounded.
   b) If $f$ is continuous on $[0, 1]$, then $f$ is bounded.
   c) If $f$ is continuous on $\mathbb{R}$ and is bounded, then $f$ attains its supremum.

6. [#15.12] Construct a function $f$ with the property that there are sequences $a_n$ and $b_n$ converging to zero such that $f(a_n)$ converges to zero but $f(b_n)$ is unbounded.

   Does there exist such a function $f$ that is continuous at $x = 0$?

7. [#15.15] Let $f(x) := x^2 + 4x$. Clearly $\lim_{x \to 0} f(x) = 0$. Assuming that $0 < \epsilon < 4$, how small must $\delta$ be so that $|x| < \delta$ implies that $|f(x)| < \epsilon$? Express $\delta$ as a function of $\epsilon$. 

1
Let \( f(a, n) := (1 + a)^n \), where \( a \) and \( n \) are positive.

a) For constant \( a \), how does \( f(a, n) \) behave as \( n \to \infty \)? For constant \( n \), how does \( f(a, n) \) behave as \( a \to 0 \)?

b) Let \( L \geq 1 \) be a given real number. Prove that there exists a sequence \( a_n \to 0 \) and \( f(a_n, n) \to L \) as \( n \to \infty \). In other words, depending on the choice of \( a_n \), \( f \) may approach any value.

Prove that there exists \( x \in [1, 2] \) such that \( x^5 + 2x + 5 = x^4 + 10 \).

Given any real number \( c > 0 \), prove there is an \( x > 0 \) such that \( x^{17} + 8x^2 = c \).

Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous (real-valued) function that is continuous at \( x = a \). If \( f(a) > 0 \), show there is an interval \( J := \{ x \in \mathbb{R} | |x - a| < \delta \} \) so that if \( x \in J \), then \( f(x) > f(a)/2 \).

Prove that any (real) polynomial whose degree is odd must have at least one real root.

Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function with the property:

\[
f(x + y) = f(x) + f(y)
\]

for all real \( x \) and \( y \)

and let \( c := f(1) \)

a) Show that \( f(0) = 0 \).

b) Show that \( f(-x) = -f(x) \) for all real \( x \).

c) If \( k \) is a positive integer show that \( f(kx) = kf(x) \) for all \( x \).

d) If \( k \) and \( n \) are positive integers, show that \( f(k) = kf(1) = kc \) and \( f(1/n) = c/n \).

e) If \( x = p/q \) is a rational number, show that \( f(x) = cx \).

f) If \( x \) is a real number, show that \( f(x) = cx \).

Say \( g : \mathbb{R} \to \mathbb{R} \) is a continuous function with the property:

\[
g(x + y) = g(x)g(y)
\]

for all real \( x \) and \( y \).

What can you conclude about \( g \)?
**Bonus Problem**

[Please give your solutions directly to Professor Kazdan]

1-B The number $e$ is defined as

$$e = 1 + 1 \cdot \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$ 

Prove that $e$ is not a rational number by the following steps.

a) Show that $2 < e < 3$. So $e$ is definitely not an integer.

b) By contradiction, say $e = \frac{p}{q}$, where $p$ and $q$ are positive integers with $q \geq 2$. Show that

$$e q! = N + \frac{c}{q + 1},$$

where $N$ is an integer and $0 < c < e$. Thus, conclude that $\frac{c}{q+1}$ must be an integer.

c) Then show that this contradicts $e < 3$ and $q + 1 \geq 3$.

[Last revised: October 13, 2018]