Directions: This exam has 7 problems (15 points each). To receive full credit your solution must be clear and correct. No fuzzy reasoning. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 × 5 with notes on both sides. Please box your answers. No fuzzy reasoning. You have 1 hour 20 minutes. Closed book.

1. A large number, \(N\), of people are subjected to a blood test, the result of which is either “positive” or “negative”. It can be given in two ways:
   i). Each person can be tested separately, so \(N\) tests are required.
   ii). The blood samples of \(k\) persons can be pooled and analyzed together. If this test is negative, then one test suffices for the \(k\) people, while if the test is positive, each of the \(k\) people must be tested separately so \(k + 1\) tests are required for the \(k\) people.
   Assume that the probability, \(p\), that a test is positive is the same for all people and that these events are all independent.
   a). Find the probability that the test for a pooled sample of \(k\) people will be positive.
   b). What is the expected value of the number of tests necessary under plan ii)? [Assume that \(N\) is divisible by \(k\)].

2. Write a Perl (or Maple) script that uses the “Monte Carlo” method for estimating the area of the ellipse
   \[
   \frac{x^2}{9} + \frac{y^2}{16} \leq 1.
   \]
   This ellipse lies inside the box \(Q = \{(x, y) : |x| \leq 3, |y| \leq 4\}\). The Monte Carlo method says to pick many points at random in the box \(Q\). The area of the ellipse will correspond to the percentage of the points that in the ellipse.
   Remark: In case it helps, here is an unrelated perl script.

```perl
#!/usr/bin/perl -w
#-------------------------- What This Does ----------------------------
# If you toss a coin at random \(N\) times, how many heads do you get?
#---------------------------- Main Program ----------------------------
$N = 10000;       # number of tosses
$heads = 0;       # initialize

for ($k=0; $k<$N; $k++) {
    $toss = int(rand(2));
    if ($toss == 0) {$heads = $heads +1;}
    if (($k % 1000) == 0) {print "In $k tosses there were $heads heads.\n";}
}
print "\n Summary: In $N tosses there were $heads heads.\n";
```
3a). Find a linear map of the plane, \( A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that does the following transformation of the letter \( \mathbf{F} \) (here the smaller \( \mathbf{F} \) is transformed to the larger one):

\[
\begin{array}{c}
\text{smaller } \mathbf{F} \\
\text{transformed to}
\end{array}
\]

b). Find a linear map of the plane that inverts this map, that is, it maps the larger \( \mathbf{F} \) to the smaller.

4. (see the graph on the right)
   a). If the horizontal axis is \( x \) and the vertical axis is \( y \), find the equation for \( y \) as a function of \( x \)?
   b). If the horizontal axis is \( x \) and the vertical axis is \( \log y \), find the equation for \( y \) as a function of \( x \).
   c). If the horizontal axis is \( \log x \) and the vertical axis is \( \log y \), find \( y \) as a function of \( x \).

5. A multinational company has branches in the U.S., Japan, and Europe. In 1990, it had assets of $5 million: $3 million are in the U.S. and $2 million in Europe. Each year \( \frac{1}{2} \) the U.S. money stays home, \( \frac{1}{4} \) goes to both Japan and Europe. For Japan and Europe, \( \frac{1}{2} \) stays home and \( \frac{1}{2} \) is sent to the U.S.
   a). Find the transition matrix of this Markov chain.
   b). Find the limiting distribution of the $5 million as the world ends.

6. Say you seek a parabola of the special form \( y = a + bx^2 \) to pass through the three data points \((-1, 2), (0, 0), (2, 3)\).
   a). Write the (over-determined) system of equations you would like to solve ideally.
   b). Using the method of least squares write the normal equations for the coefficients \( a, b \).
   c). Explicitly find the coefficients \( a, b \).
7. Discuss which airline has the better “On Time” record.

<table>
<thead>
<tr>
<th>Destination</th>
<th>Northern Airline</th>
<th>Travel West</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td># arrivals</td>
<td># on time</td>
</tr>
<tr>
<td>Phoenix</td>
<td>200</td>
<td>190</td>
</tr>
<tr>
<td>Seattle</td>
<td>2000</td>
<td>1700</td>
</tr>
<tr>
<td>Total</td>
<td>2200</td>
<td>1890</td>
</tr>
</tbody>
</table>

[We are seeking a quantitative response, not an intuitive one. Since you don’t have calculators, approximate calculations will be adequate.]