DIRECTIONS: Part A has 4 short questions (10 points each), Part B has 3 traditional problems (20 points each),
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour
20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Four shorter problems, 10 points each [total: 40 points]

A-1. Say you have \( k \) linear algebraic equations in \( n \) variables; in matrix form we write \( AX = Y \).
Give an explicit counterexample for each of the following assertions.

a) If \( n = k \) there is always at most one solution.

b) If \( n > k \) you can always solve \( AX = Y \).

c) If \( n < k \) the only solution of \( AX = 0 \) is \( X = 0 \).

A-2. Let \( S \) and \( T \) be linear spaces and \( L : S \to T \) be a linear map. Say \( v_1 \) and \( v_2 \) are (distinct!)
solutions of the equations \( Lx = y_1 \) while \( w \) is a solution of \( Lx = y_2 \). Answer the following in
terms of \( v_1, v_2, \) and \( w \).

a) Find some solution of \( Lx = 2y_1 - 3y_2 \).

b) Find another solution (other than \( w \)) of \( Lx = y_2 \).

A-3. Find a \( 2 \times 2 \) matrix \( A \) that in the standard basis transforms the larger \( \mathbf{F} \) (on the left) to the smaller.

A-4. Let \( X \) and \( Y \) be vectors in \( \mathbb{R}^n \). If the Pythagorean Theorem holds:

\[
\|X + Y\|^2 = \|X\|^2 + \|Y\|^2,
\]

show that \( X \) and \( Y \) are orthogonal.
PART B: Three traditional problems, 20 points each [60 points]

B-1. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs. One has data on which location the cars are returned daily:

- **Rented at Airport**: 5% are returned to the City and 20% to the Suburbs. The rest are returned to the Airport.
- **Rented in City**: 10% are returned to Airport, 10% returned to Suburbs.
- **Rented in Suburbs**: 20% are returned to the Airport and 5% to the City.

If initially there are 20 cars at the Airport, 65 in the city, and 15 in the suburbs, what is the long-term distribution of the cars?

B-2. A friend is tested for a relatively rare cancer that occurs in only 1 out of every 10,000 people his age. The test is accurate in the sense that:

- 10% of those who do not have the cancer still test positive (false positives)
- 2% of those who have the cancer test negative (false negatives).

If your friend tests positive, what is the likelihood that he has the cancer?

*There is no need to do any arithmetic to “simplify” your result. I’ll assume you can do that.*

B-3. [Making a Triangle]

a) First a calculation. Choose independently two numbers $x$ and $y$ at random from the unit interval $0 \leq t \leq 1$. This defines a random point $(x, y)$ in the unit square. Find the probability that this point is in the set

$$\{ x \leq y, \quad x \leq 1/2, \quad y - x \leq 1/2, \quad \text{and} \quad 1 - y \leq 1/2 \}.$$ 

b) You want to make a triangle whose sides have length $a$, $b$, and $c$ with $a + b + c = 1$. Which numbers will work? If any side has length greater than 1/2 then the triangle can’t close up. Thus we need

$$a \leq 1/2, \quad b \leq 1/2, \quad \text{and} \quad c \leq 1/2, \quad \text{along with} \quad a + b + c = 1.$$ 

To determine the pieces $a$, $b$, and $c$, choose two numbers $u$ and $v$ at random from the unit interval $0 \leq t \leq 1$. There are two cases: if $u \leq v$: $0 \underline{u} \underline{v} 1$, then let $a = u$, $b = v - u$, and $c = 1 - v$. The other case, $v \leq u$, we similarly let $a = v$, $b = u - v$, and $c = 1 - u$.

Now use part a) to determine the probability that the three pieces can be used to form a triangle.