Directions: Part A has 4 short questions (10 points each), Part B has 3 traditional problems (20 points each).
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 × 5 with notes on both sides.

PART A: Four shorter problems, 10 points each [total: 40 points]

A-1. Say you have \( k \) linear algebraic equations in \( n \) variables; in matrix form we write \( AX = Y \).
Give an explicit counterexample for each of the following assertions.

a) If \( n = k \) there is always at most one solution.
Solution: \( n = k = 2 \). Two examples:
\[
\begin{align*}
0x_1 + 0x_2 &= 0 \\
1x_1 + 1x_2 &= 0 \\
0x_1 + 0x_2 &= 0 \\
2x_1 + 2x_2 &= 0
\end{align*}
\]

b) If \( n > k \) you can always solve \( AX = Y \).
Solution: \( n = 2, k = 1 \): \( 0x_1 + 0x_2 = 3 \)
\[
\begin{align*}
n = 3, k = 2 : \\
x_1 + x_2 + x_3 &= 1 \\
x_1 + x_2 + x_3 &= 2
\end{align*}
\]

c) If \( n < k \) the only solution of \( AX = 0 \) is \( X = 0 \).
Solution:
\[
\begin{align*}
n = 1, k = 2 & : 0x_1 = 0 \\
0x_1 &= 0 \\
n = 2, k = 3 & : x_1 + x_2 = 0 \\
2x_1 + 2x_2 &= 0 \\
3x_1 + 3x_2 &= 0
\end{align*}
\]

A-2. Let \( S \) and \( T \) be linear spaces and \( L : S \rightarrow T \) be a linear map. Say \( v_1 \) and \( v_2 \) are (distinct!) solutions of the equations \( Lx = y_1 \) while \( w \) is a solution of \( Lx = y_2 \). Answer the following in terms of \( v_1, v_2, \) and \( w \).

a) Find some solution of \( Lx = 2y_1 - 3y_2 \).
Solution: \( x = 2v_1 = 3w \)

b) Find another solution (other than \( w \)) of \( Lx = y_2 \).
Solution: \( x = w + (v_1 - v_2) \)
A-3. Find a $2 \times 2$ matrix $A$ that in the standard basis transforms the larger $F$ (on the left) to the smaller.

Solution: Say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note $A : (-10,0) \rightarrow (0,5)$ and $A : (0,1) \rightarrow (1,0)$. Thus

\[
\begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -10 \\ 0 \end{pmatrix} = \begin{pmatrix} -10a \\ -10c \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}
\]

so $a = 0$, $b = 1$, $c = -1/2$, and $d = 0$; that is, $A = \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$.

Alternate: The map $B$ from the small $F$ to the larger one is simpler: $B = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$ and $A = B^{-1}$.

A-4. Let $X$ and $Y$ be vectors in $\mathbb{R}^n$. If the Pythagorean Theorem holds:

\[
\|X + Y\|^2 = \|X\|^2 + \|Y\|^2,
\]

show that $X$ and $Y$ are orthogonal.

Solution:

\[
\|X + Y\|^2 = \langle X + Y, X + Y \rangle = \|X\|^2 + 2\langle X, Y \rangle + \|Y\|^2
\]

so $\langle X, Y \rangle = 0$, that is, $X$ and $Y$ are orthogonal.

Part B: Three traditional problems, 20 points each [60 points]

B-1. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs. One has data on which location the cars are returned daily:

- **Rented at Airport:** 5% are returned to the City and 20% to the Suburbs. The rest are returned to the Airport.
• **Rented in City**: 10% are returned to Airport, 10% returned to Suburbs, the rest returned to the City.

• **Rented in Suburbs**: 20% are returned to the Airport and 5% to the City, the rest returned to the Suburbs.

If initially there are 20 cars at the Airport, 65 in the city, and 15 in the suburbs, what is the long-term distribution of the cars?

**Solution:**

\[ A_{k+1} = .75A_k + .10C_k + .20S_k \]
\[ C_{k+1} = .05A_k + .80C_k + .05S_k \]
\[ S_{k+1} = .20A_k + .10C_k + .75S_k \]

so the transition matrix is

\[ T = \begin{pmatrix} .75 & .10 & .20 \\ .05 & .80 & .05 \\ .20 & .10 & .75 \end{pmatrix} \]

Let \( P = \begin{pmatrix} A \\ C \\ S \end{pmatrix} \), with \( A + C + S = 1 \) be the long-term distribution. Then \( TP = P \), so we need to solve \((T - I)P = 0\), that is

\[-.25A + .10C + .20S = 0\]
\[.05A - .20C + .05S = 0\]
\[.20A + .10C - .25S = 0\]

along with \( A + C + S = 1 \). We find \( A = S = 2C \), so \( P = \begin{pmatrix} .40 \\ .20 \\ .40 \end{pmatrix} \). Since initially there were 20 + 65 + 15 = 100 cars, the long-term distribution is Airport: 40, City: 20, and Suburbs: 40 cars.

B-2. A friend is tested for a relatively rare cancer that occurs in only 1 out of every 10,000 people his age. The test is accurate in the sense that:

• 10% of those who do not have the cancer still test positive (false positives)
• 2% of those who have the cancer test negative (false negatives).

If your friend tests positive, what is the likelihood that he has the cancer?

[There is no need to do any arithmetic to “simplify” your result. I’ll assume you can do that.]

**Solution:**

\[ P(C|+) = \frac{P(+|C)P(C)}{P(+)} = \frac{P(+|C)P(C)}{P(+|C)P(C) + P(+|\neg C)P(\neg C)} \]
\[ = \frac{.98}{.98 + 999.9} \approx \frac{1}{1,000} \]
a) First a calculation. Choose independently two numbers \( x \) and \( y \) at random from the unit interval \( 0 \leq t \leq 1 \). This defines a random point \((x, y)\) in the unit square. Find the probability that this point is in the set

\[
S = \{ \quad x \leq y, \quad \text{and} \quad x \leq 1/2, \quad \text{and} \quad y - x \leq 1/2, \quad \text{and} \quad 1 - y \leq 1/2 \quad \}
\]

**Solution:**
The region \( S \) is the triangle in the figure. Since its area is \(1/8\) of the area of the square, the probability that the point is in \( S \) is \(.125\).

b) You want to make a triangle whose sides have length \( a \), \( b \), and \( c \) with \( a + b + c = 1 \). Which numbers will work? If any side has length greater than \( 1/2 \) then the triangle can’t close up. Thus we need

\[
a \leq 1/2, \quad b \leq 1/2, \quad \text{and} \quad c \leq 1/2, \quad \text{along with} \quad a + b + c = 1.
\]

To determine the pieces \( a \), \( b \), and \( c \), choose two numbers \( u \) and \( v \) at random from the unit interval \( 0 \leq t \leq 1 \). There are two cases: if \( u \leq v \): 0 \( \xrightarrow{u} \) \( \xrightarrow{v} \) 1, then let \( a = u \), \( b = v - u \), and \( c = 1 - v \). The other case, \( v \leq u \), we similarly let \( a = v \), \( b = u - v \), and \( c = 1 - u \).

Now use part a) to determine the probability that the three pieces can be used to form a triangle.

**Solution:** The case \( u \leq v \) exactly describes the set \( S \) in part a). The other case, \( v \leq u \), is just the reflection of \( S \) across the diagonal so it has the same area as \( S \). Thus, the probability that the pieces can be used to form a triangle is \( 1/8 + 1/8 = 1/4 = .25 \).