Linear Algebra: An Outline with Examples

Linear Space (= Vector Space): cv, $v + w$. Letting $c = 0$ shows that a linear space must always have the 0 vector.

Examples: $\mathbb{R}^2$, R^n

Polynomials of degree at most two: $\mathcal{P}_2$, $p(x) = a_0 + a_1 x + a_2 x^2$

The straight line: $\{(x, y) \in \mathbb{R}^2\, x + y = 0\}$ is a linear space. It is a *linear subspace* of $\mathbb{R}^2$.

The straight line: $\{(x, y) \in \mathbb{R}^2\, x + y = 1\}$ is *not* a linear space since it does not have the zero vector.

The upper half plane $\{(x, y) \in \mathbb{R}^2 \,:\, y \geq 0\}$ is not a linear space since it contains $V = (0,\, 1)$ but it does not contain $(0,\, -1\ (c = -1)$

Words: span, linearly independent, basis, dimension, linear subspace.

Picking Coordinates Idea: adapt them to the problem.

Linear Maps $L : V \to W$ Linear maps have two properties:

$$L(cX) = cL(X), \qquad L(X + Y) = L(X) + L(Y)$$

Letting $c = 0$ implies in particular that $L(0) =$, it maps the origin in V to the origin in W.
The following maps from $\mathbb{R}$ to $\mathbb{R}$ are *not* linear:

$$f(x) = x^2, \qquad g(x) = \sqrt{x}, \qquad \frac{1}{x}$$

The formulas

$$(x + y)^2 = x^2 + y^2, \qquad \sqrt{x + y} = \sqrt{x} + \sqrt{y}, \quad \text{and} \quad \frac{1}{x + y} = \frac{1}{x} + \frac{1}{y}$$

are not true. These may have annoyed you at an earlier point of your life (they may still annoy you).

Examples: $A : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{aligned} ax_1 + bx_2 &= y_1 \\ cx_1 + dx_2 &= y_2 \end{aligned}$$

The letter **F**:
`https://www.math.upenn.edu/~kazdan/312S13/Notes/F1.pdf`

Example:
The simplest matrices are square diagonal matrices.

Examples 0: The *zero matrix* $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which maps every point to 0.
The *identity matrix* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which maps every point to itself

Example 1: $R =$ reflection across the line $x_2 = 0$. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Find the eigenvalues and corresponding eigenvectors. $\lambda_1 = 1$, $\lambda_2 = -1$, ${}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example 2: $R =$ reflection across the line $x_1 = x_2$. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find the eigenvalues and corresponding eigenvectors. $\lambda_1 = 1$, $\lambda_2 = -1$, $V_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $V_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

`https://www.math.upenn.edu/~kazdan/312S13/Notes/3x2.pdf` Linear maps from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Example 3: If $x_{n+2} = x_{n+1} + x_n$, with $x_0 = 0$ and $x_1 = 1$, find a formula for x_n. These are the *Fibonacci numbers*: 0, 1, 2, 3, 5, 8, 13, 21, 34, ... These arise in a number of situations, such as the growth of plants and trees.

To begin we rewrite this two-step process as a one-step system of linear equations. Let $u_n = x_n$, $v_n = u_{n+1}$. Then

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \qquad \text{so} \qquad U_{n+1} = AU_n, \qquad U_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Consequently, $U_n = A^n U_0$. If A were a diagonal matrix, then A^n would be simple to compute. Here is where eigenvalues and eigenvectors help us.

Find the eigenvalues of A: $0 = \det(A - \lambda I) = \lambda^2 - \lambda - 1$, so $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Corresponding eigenvectors: $W_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$, $W_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$.

Write U_0 in this basis: $U_0 = aW_1 + bW_2$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + b \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

so $a + b = 0$ and $a\lambda_1 + b\lambda_2 = 1$. Thus $U_0 = \frac{W_1 - W_2}{\lambda_1 - \lambda_2}$.

Consequently $U_{100} = \frac{\lambda_1^{100} W_1 - \lambda_2^{100} W_2}{\lambda_1 - \lambda_2}$.

For the Fibonacci number x_{100} we want the first component of U_{100}

$$x_{100} = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2}.$$

Example $A : \mathbb{R}^2 \to \mathbb{R}^3$, $B : \mathbb{R}^3 \to \mathbb{R}^2$, so $BA : \mathbb{R}^2 \to \mathbb{R}^2$ $AB : \$^3 \to \mathbb{R}^3$.

Words: homogeneous equation, inhomogenuous equation, one-to-one (injective), onto (surjective), invertible (inverse map, bijective, isomorphism)).

Remark: Notable formula: $(BA)^{-1} = A^{-1}B^{-1}$. This formula has almost nothing to do with matrices; it is valid for all invertible maps.

$D : \mathcal{P}_\in \to \mathcal{P}_\in$ The derivative: $D(a + bx + cx^2) = b + 2cx$

Note: $D^2(a + bx + cx^2) = D(b + 2cx) = 2c$ so $D^3(a + bx + cx^2) = 0$.

https://www.math.upenn.edu/~kazdan/312S14/Notes/kernel-image.pdf A summary of basic facts.

Example: POLYNOMIAL INTERPOLATION. $L : \mathcal{P}_n \to \mathbb{R}^{n+1}$. Given the distinct real numbers x_0, x_1, $\ldots, x_n$, let $L(p)$ the the value of the polynomial $p(x)$ at these points, so

$$L(p) = (p(x_0,\, p(x_1), \ldots, p(x_n)).$$

For instance, if $n = 2$, say we want a quadratic polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ that passes through the three points (x_0, y_0), (x_1, y_1), (x_2, y_2), so

$$p(x_0) = y_0, \qquad , p(x_1) = y_1 \quad \text{and} \quad p(x_2) = y_2.$$

These are three linear equations in the three unknowns a_0, a_1, and a_2. Is there always a solution? If so, is it unique?

Example: Let $L : \mathcal{P}_k \to \mathcal{P}_k$ be $L(p) = p' + 2p$ (here p' is the first derative). Given a quadratic polynomial q can one always find a quadratic polynomial p so that $Lp =$, that is, $p' + 2p = q$? If so, is it unique?

https://www.math.upenn.edu/~kazdan/210S19/Notes/Maple/ Maple Examples

https://www.math.upenn.edu/~kazdan/504/la.pdf Large collection of Linear Algebra Problems