Typical Linear Programming Examples

REFERENCE: Gilbert Strang, *Linear Algebra and Its Applications*, 4th Edition, Brooks/Cole 2006 [Chapter 8]

Production Planning Suppose General Motors makes a profit of $200 on each Chevrolet, $300 on each Buick, and $500 on each Cadilac. These get 20, 17, and 14 miles per gallon, respectively. Congress passed a law that the average car must get 18. The plant can assemble a Chevrolet in one minute, a Buick in 2 minutes, and a Cadilac in 3 minutes.

What is the maximum profit, P, in 8 hours (480 minutes)?

Let x number of Chevrolets, y Buicks, z Cadilacs.

SUMMARY: Maximize $P = 200x + 300y + 500z$

subject to the constraints:

$$20x + 17y + 14z \geq 18(x + y + z), \quad x + 2y + 3z \leq 480,$$
$$x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

Portfolio Selection Say federal bonds pay 5% interest, municipals pay 6%, amd junk bonds pay 9%. We buy dollar amounts of x, y, z and have \$100,000. The problem is to maximize the interest with two constraints:

(i) No more that \$20,000 can be invested in junk bonds.

(ii) The portfolio's average quality cannot be lower than municipals, so $x \geq z$.

SUMMARY: Maximize $5x + 6y + 9z$ subject to

$$x+y+z \leq 100,000, \qquad z \leq 20,000, \qquad z \leq x, \qquad x,\, y,\, z \geq 0.$$

REMARK: There is a simple way to change the first three inequalities like $x+y+z \leq 100,000$ into equations: just introduce the three differences as *slack variable*

$$u := 100,000 - x - y - z, \qquad v := 20,000 - z, \qquad w := x - z$$

with the constraints $u \geq 0$, $v \geq 0$, and $w \geq 0$.

Then we have three *equality* constraints

$$x+y+z+u = 100,000, \qquad z+v = 20,000, \qquad x-z-v = 0$$

and the *inequalities*

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad u \geq 0, \quad v \geq 0, \quad w \geq 0.$$

This has more variables but is concetpually easier to manage. In the language of linear algebra, the problem now has the form

$$\textbf{Maximize } \vec{c} \cdot \vec{x} \quad \textbf{subject to} \quad A\vec{x} = \vec{b} \textbf{ and } \vec{x} \geq 0.$$

Here $\vec{x} := (x, y, z, u, v, w)$ and $\vec{c} := (5, 6, 9, 0, 0, 0)$. Note that the zeroes in $\vec{c}$ correspond to the coefficients of the slack variables in the income (which we want to maximize).