Distance From a Point To a Line

1. Find the distance from the point $P = (2, 3)$ to the straight line $\mathcal{L} := \{(x, y) \in \mathbb{R}^2 : x + 2y = 0\}$.

   Solution: Method 1. Let $Q = (x, y)$ be a point on the line and $U = (1, 2)$. Then the equation of the line means that $\langle Q, U\rangle = 0$ so the vector U is orthogonal to the points on the line.
$$\mathcal{L}_0 = \{Q \in \mathbb{R}^2 : \langle Q, U\rangle = 0\}. \tag{1}$$
   To find the distance from P to $\mathcal{L}$, we find the orthogonal projection of P onto a *unit vector* $\hat{U}$, in the direction of U, so $\hat{U} = U/\sqrt{5}$. Now
$$P = c\hat{U} + W, \qquad \text{where} \qquad W \perp \hat{U}.$$
   Taking the inner product of both sides with $\hat{U}$ we find
$$\langle P, \hat{U}\rangle = c\langle \hat{U}, \hat{U}\rangle, \qquad \text{so} \qquad c = \frac{\langle P, \hat{U}\rangle}{\|\hat{U}\|^2} = \frac{8}{\sqrt{5}}.0$$
   The desired projection is distance is $|c| = 8/\sqrt{5}$. Note $W = P - c\hat{U} = \frac{1}{5}(2, -1)$.

   Method 2 (calculus). We think of the points on line as the path of a particle at time t. Let $V = (2, -1)$ (think of V as a velocity vector). Then
$$\mathcal{L}_0 = \{Q \in \mathbb{R}^2 : Q(t) = tV \quad \text{for some} \quad t \in \mathbb{R}.$$
   We want to t_0 to minimize $h(t) := \|P - Q(t)\|^2$, so
$$0 = h'(t) = 2\langle P - Q(t), -Q'(t)\rangle = 2\langle P - tV, -V\rangle$$
   which gives $t_0 = \langle P, V\rangle/\|V\|^2 = 1/5$ and $t_0V = \frac{1}{5}(2, -1$ is the point on the line that is closest to P. This is the same as the point W found above. Also $P - t_0V = \frac{8}{5}(1, 2) = c\hat{U}$ found above. In particular, the distance from P to the line is
$$\|P - t_0V\| = 8/\sqrt{5}.$$

2. Find the distance from the point $P = (2, 3)$ to the straight line $\mathcal{L}_1 := \{(x, y) \in \mathbb{R}^2 : x + 2y = 3\}$. The change from the previous example is that this line does not pass through the origin.

   Solution: Method 1. Let $Q = (x, y)$ be a point on the line and $U = (1, 2)$. Then the equation of the line means that $\langle Q, U\rangle = 3$. Pick a specific point Q_0 on the line, say $Q_0 = (1, 1)$ (in the previous example we let $Q_0 = (0, 0)$). Then the equation of the line is $\langle Q, U\rangle = \langle Q_0, U\rangle$, That is, $\langle Q - Q_0, U\rangle = 0$, be so the vector U is orthogonal to $Q - Q_0$.

To find the distance from P to $\mathcal{L}_1$ we find the orthogonal projection of $P - Q_0$ onto the *unit vector* $\hat{U}$, in the direction of U, so $\hat{U} = U/\sqrt{5}$. Now

$$P - Q_0 = c\hat{U} + W, \qquad \text{where} \qquad W \perp \hat{U}.$$

Taking the inner product of both sides with $\hat{U}$, since $P - Q_0 = (1, 2)$ we find

$$\langle P - Q_0,\, \hat{U}\rangle = c\langle \hat{U},\, \hat{U}\rangle, \qquad \text{so} \qquad c = \frac{\langle P - Q_0,\, \hat{U}\rangle}{\|\hat{U}\|^2} = \frac{5}{\sqrt{5}} = \sqrt{5}.$$

The desired projection is distance is $|c| = \sqrt{5}$. Note in this case, by chance choice of Q_0, that $W = (P - Q_0) - c\hat{U} = 0$.

Method 2 (calculus). This works for a straight line in $\mathbb{R}^n$.

Let $Z \in \mathbb{R}^n$ be a point and $\mathcal{L}$ the straight line

$$\mathcal{L} = \{X \in \mathbb{R}^n \mid X = X_0 + tV\},$$

where $X_0 \in \mathbb{R}^n$ is a specified point, $V \in \mathbb{R}^n$ is a unit vector, and $t \in \mathbb{R}$ (think of $X(t)$ as the position of a particle at time t). Since adding a multiple of V to X_0 does not change the line, we may assume that X_0 is orthogonal to V, so $\langle X_0,\, V\rangle = 0$. Then for fixed V and various X_0 define parallel lines whose distance from the origin is $\|X_0\|$.

Compute the (Euclidean) distance from Z to the line.

SOLUTION: We minimize

$$\varphi(t) = \|Z - (X_0 + tV)\|^2.$$

At a minimum,

$$0 = \varphi'(t) = 2\langle Z - (X_0 + tV),\, -V\rangle.$$

Because $\|V\| = 1$ then $t = \langle Z - X_0,\, V\rangle = \langle Z,\, V\rangle$ and hence

$$\begin{aligned}\text{Distance}^2(Z,\, \mathcal{L}) &= \|Z - X_0 - \langle Z,\, V\rangle V\|^2 \\ &= \|Z - X_0\|^2 - \langle Z,\, V\rangle^2.\end{aligned}$$

Because $\langle Z,\, V\rangle V$ is just the orthogonal projection of Z into $\mathcal{L}$, this formula is also a geometrically obvious consequence of the Pythagorean Theorem.