Problem Set 4

DUE: In class Tues. Feb. 19 [Late papers will be accepted until 1:00 PM Wed].

Lots of problems this week. Fortunately a number of them are short – but don’t wait until Monday night!

Most of these problems should be a review of the basic linear algebra of Math 240, but emphasizing thinking of a system of linear equations as a linear mapping. They should be very short. In class on we’ll discuss this more.

Problems

1. If $A$ is a $5 \times 5$ matrix with $\det A = -1$, compute $\det(-2A)$.

2. Consider the system of equations

$$
\begin{align*}
x + y - z &= a \\
x - y + 2z &= b.
\end{align*}
$$

a) Find the general solution of the homogeneous equation, so $a = b = 0$.

b) A particular solution of the inhomogeneous equations when $a = 1$ and $b = 2$ is $x = 1, y = 1, z = 1$. Find the most general solution of the inhomogeneous equations.

c) Find some particular solution of the inhomogeneous equations when $a = -1$ and $b = -2$.

d) Find some particular solution of the inhomogeneous equations when $a = 3$ and $b = 6$.

[Remark: After you have done part a), it is possible immediately to write the solutions to the remaining parts.]

3. Solve the given system – or show that no solution exists:

$$
\begin{align*}
x + 2y &= 1 \\
3x + 2y + 4z &= 7 \\
-2x + y - 2z &= -1
\end{align*}
$$

4. Say you have $k$ linear algebraic equations in $n$ variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following.

a) If $n = k$ there is always at most one solution.

b) If $n > k$ you can always solve $AX = Y$.  

1
c) If \( n > k \) the homogeneous equation \( AX = 0 \) has at least one solution \( X \neq 0 \).

d) If \( n < k \) then for some \( Y \) there is no solution of \( AX = Y \).

e) If \( n < k \) the only solution of \( AX = 0 \) is \( X = 0 \).

5. Let \( A : \mathbb{R}^n \rightarrow \mathbb{R}^k \) be a real matrix, not necessarily square. If two rows of \( A \) are the same, show that \( A \) is not onto by finding a vector \( y = (y_1, \ldots, y_k) \) that is not in the image of \( A \). [Hint: This is a mental computation if you write out the equations \( Ax = y \) explicitly.]

6. Let \( A : \mathbb{R}^n \rightarrow \mathbb{R}^k \) be a real matrix, not necessarily square. If two columns of \( A \) are the same, show that \( A \) is not one-to-one by finding a vector \( x = (x_1, \ldots, x_n) \) that satisfies \( Ax = 0 \).

7. The following 2 × 2 matrices are valuable examples that may be surprising

\[
P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = PR = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Geometrically, \( P \) is an orthogonal projection onto the \( x_1 \) axis, that is, if \( X = (x_1, x_2) \in \mathbb{R}^2 \) is a (column) vector in the plane, then \( PX \) is its orthogonal projection onto the \( x_1 \) axis. Similarly, \( R \) is a rotation by 90 degrees clockwise.

Compute (and interpret geometrically):

\( P^2, \ P^3, \ R^2, \ R^3, \ R^4, \ PR, \ RP, \ C^2, \ CP, \ PC \).

8. Let \( A \) and \( B \) be \( n \times n \) matrices with \( AB = 0 \). Give a proof or counterexample for each of the following.

a) Either \( A = 0 \) or \( B = 0 \) (or both).

b) \( BA = 0 \)

c) If \( \det A = -3 \), then \( B = 0 \).

d) If \( B \) is invertible then \( A = 0 \).

e) There is a vector \( V \neq 0 \) such that \( BAV = 0 \).

9. Let \( A \) be a 4 × 4 matrix with determinant 7. Give a proof or counterexample for each of the following.

a) For some vector \( b \) the equation \( Ax = b \) has exactly one solution.

---

1The computer graphics examples in [https://www.math.upenn.edu/~kazdan/320F18/notes/Maple/F1.pdf](https://www.math.upenn.edu/~kazdan/320F18/notes/Maple/F1.pdf) may also be illuminating.
b) For some vector \( \mathbf{b} \) the equation \( A \mathbf{x} = \mathbf{b} \) has infinitely many solutions.

c) For some vector \( \mathbf{b} \) the equation \( A \mathbf{x} = \mathbf{b} \) has no solution.

d) For all vectors \( \mathbf{b} \) the equation \( A \mathbf{x} = \mathbf{b} \) has at least one solution.

10. a) Find a \( 2 \times 2 \) matrix that rotates the plane by +45 degrees (+45 degrees means 45 degrees counterclockwise).

b) Find a \( 2 \times 2 \) matrix that rotates the plane by +45 degrees followed by a reflection across the horizontal axis.

c) Find a \( 2 \times 2 \) matrix that reflects across the horizontal axis followed by a rotation the plane by +45 degrees.

d) Find a matrix that rotates the plane through +60 degrees, keeping the origin fixed.

e) Find the inverse of each of these maps.

11. Find a real \( 2 \times 2 \) matrix \( A \) (other than \( A = I \)) such that \( A^5 = I \).

12. Proof or counterexample. In these \( L \) is a linear map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), so its representation will be as a \( 2 \times 2 \) matrix.

a) If \( L \) is invertible, then \( L^{-1} \) is also invertible.

b) If \( LV = 5V \) for all vectors \( V \), then \( L^{-1}W = (1/5)W \) for all vectors \( W \).

c) If \( L \) is a rotation of the plane by 45 degrees counterclockwise, then \( L^{-1} \) is a rotation by 45 degrees clockwise.

d) If \( L \) is a rotation of the plane by 45 degrees clockwise, then \( L^{-1} \) is a rotation by 315 degrees counterclockwise.

e) The zero map (\( 0V = 0 \) for all vectors \( V \)) is invertible.

f) The identity map (\( IV = V \) for all vectors \( V \)) is invertible.

g) If \( L \) is invertible, then \( L^{-1}0 = 0 \).

h) If \( LV = 0 \) for some non-zero vector \( V \), then \( L \) is not invertible.

i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: \( L = L^{-1} \).

13. Let \( A \) be a matrix, not necessarily square. Say \( \mathbf{V} \) and \( \mathbf{W} \) are particular solutions of the equations \( A \mathbf{V} = \mathbf{Y}_1 \) and \( A \mathbf{W} = \mathbf{Y}_2 \), respectively, while \( \mathbf{Z} \neq 0 \) is a solution of the homogeneous equation \( A \mathbf{Z} = 0 \). Answer the following in terms of \( \mathbf{V} \), \( \mathbf{W} \), and \( \mathbf{Z} \).

a) Find some solution of \( A \mathbf{X} = 3 \mathbf{Y}_1 \).

b) Find some solution of \( A \mathbf{X} = -5 \mathbf{Y}_2 \).

c) Find some solution of \( A \mathbf{X} = 3 \mathbf{Y}_1 - 5 \mathbf{Y}_2 \).
d) Find another solution (other than $Z$ and 0) of the homogeneous equation $AX = 0$.

e) Find two solutions of $AX = Y_1$.

f) Find another solution of $AX = 3Y_1 - 5Y_2$.

g) If $A$ is a square matrix, then $\det A = ?$

h) If $A$ is a square matrix, for any given vector $W$ can one always find at least one solution of $AX = W$? Why?

14. Let $R$, $M$, and $N$ be linear maps from the (two dimensional) plane to the plane given in terms of the standard $i, j$ basis vectors by:

\[ Ri = j, \quad Rj = -i, \quad Mi = -i, \quad Mj = j \quad Nv = -v \quad \text{for all vectors } v \]

a) Describe (pictures?) the actions of the maps $R, R^2, R^{-1}, M, M^2, M^{-1}$ and $N$.

b) Describe the actions of the maps $RM, MR, RN, NR, MN$, and $NM$ [here we use the standard convention that the map $RM$ means first use $M$ then $R$]. Which pairs of these maps commute?

c) Which of the following identities are correct—and why?

1) $R^2 = N$  
2) $N^2 = I$  
3) $R^4 = I$  
4) $R^5 = R$  
5) $M^2 = I$  
6) $M^3 = M$  
7) $MNM = N$  
8) $NMN = R$

d) Find matrices representing each of the maps $R, R^2, R^{-1}, M$, and $N$.

15. a). Find a linear map of the plane, $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that does the following transformation of the letter $F$ (here the smaller $F$ is transformed to the larger one):

![Graph showing linear transformation of letter F]

b). Find a linear map of the plane that inverts this map, that is, it maps the larger $F$ to the smaller.
16. Linear maps \( F(X) = AX \), where \( A \) is a matrix, have the property that \( F(0) = A0 = 0 \), so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

\[
F(X) = V + AX,
\]

where \( V \) is a vector. Note that \( F(0) = V \).

Find the vector \( V \) and the matrix \( A \) that describe each of the following mappings [here the light blue \( F \) is mapped to the dark red \( F \)].

![Graphs showing mappings](image)

**Bonus Problem**

[Please give solutions directly to Professor Kazdan]

1-B [How to Rotate a Mattress]. It is standard to rotate a mattress so that it wears more evenly. For this task, one needs to understand the symmetries of a mattress – which is just a rectangular box whose height, width, and length are distinct.

[As a warm-up, understand all the symmetries of a square.]

[Last revised: February 12, 2019]