Problem Set 0

Due: Never

This is rust remover. Essentially all of the ideas needed for these problems should have been covered in the prerequisite courses. No messy computations are needed – but some serious thinking may be needed. Notation: \( u_t = \frac{\partial u}{\partial t} \).

1. Let \( u(t) \) be the solution of \( u' = 3u \) with initial value \( u(0) = A > 0 \). At what time \( T \) is \( u(T) = 2A \)?

2. Let \( u(t) \) be the amount of a radioactive element at time \( t \) and say initially, \( u(0) = A > 0 \). The rate of decay is proportional to the amount present, so
\[
\frac{du}{dt} = -cu,
\]
where the constant \( c > 0 \) determines the decay rate. The half-life \( T \) is the amount of time for half of the element to decay, so \( u(T) = \frac{1}{2}u(0) \). Find \( c \) in terms of \( T \) and obtain a formula for \( u(t) \) in terms of \( T \).

3. Let \( \int_0^x f(t) \, dt = e^{\cos(3x)} + A \), where \( f \) is some continuous function. Find \( f \) and the constant \( A \).

4. a) If \( u'' + 4u = 0 \) with initial conditions \( u(0) = 1 \) and \( u'(0) = -2 \), compute \( u(t) \).
   b) Find a particular solution of the inhomogeneous equation \( u'' + 4u = 8 \).
   c) Find a particular solution of the inhomogeneous equation \( u'' + 4u = -4t \).
   d) Find a particular solution of the inhomogeneous equation \( u'' + 4u = -8 - 8t \).
   e) Find the most general solution of the inhomogeneous equation \( u'' + 4u = 8 - 8t \).
   f) If \( f(t) \) is any continuous function, use the method “variation of parameters” (look it up if you don’t know it) to find a formula for a particular solution of \( u'' + 4u = f(t) \).

5. Let \( u(t) \) be any solution of \( u'' + 2bu' + 4u = 0 \). If \( b > 0 \) is a constant, show that \( \lim_{t \to \infty} u(t) = 0 \).

6. a) If \( u'' - 4u = 0 \) with initial conditions \( u(0) = 1 \) and \( u'(0) = -2 \), compute \( u(t) \).
   b) Find a particular solution of the inhomogeneous equation \( u'' - 4u = 8 \).
   c) Find a particular solution of the inhomogeneous equation \( u'' - 4u = -4t \).
d) Find a particular solution of the inhomogeneous equation $u'' - 4u = -8 - 8t$.
e) Find the most general solution of the inhomogeneous equation $u'' - 4u = 8 - 8t$.
f) If $f(t)$ is any continuous function, use the method “variation of parameters” (look it up if you don’t know it) to find a formula for a particular solution of $u'' - 4u = f(t)$.

7. Say $w(t)$ satisfies the differential equation

$$aw''(t) + bw' + cw(t) = 0, \tag{1}$$

where $a$ and $c$, are positive constants and $b \geq 0$. Let $E(t) = \frac{1}{2}[aw'^2 + cw^2]$.

a) Without solving the differential equation, show that $E'(t) \leq 0$.

b) Use this to show that If you also know that $w(0) = 0$ and $w'(0) = 0$, then $w(t) = 0$ for all $t \geq 0$.

c) [Uniqueness] Say the functions $u(t)$ and $v(t)$ both satisfy the same equation (1) and also $u(0) = v(0)$ and $u'(0) = v'(0)$. Show that $u(t) = v(t)$ for all $t \geq 0$.

8. Say $u(x, t)$ has the property that $\frac{\partial u}{\partial t} = 2$ for all points $(x, t) \in \mathbb{R}^2$.

a) Find some such function $u(x, t)$.

b) Find the most general such function $u(x, t)$.

c) If $u(x, 0) = \sin 3x$, find $u(x, t)$.

d) If instead $u$ satisfies $\frac{\partial u}{\partial t} = 2xt$, still with $u(x, 0) = \sin 3x$, find $u(x, t)$.

9. Say $u(x, t)$ has the property that $\frac{\partial u}{\partial t} = 3u$ for all points $(x, t) \in \mathbb{R}^2$.

a) Find some such function – other than the trivial $u(x, t) \equiv 0$.

b) Find the most general such function.

c) If $u(x, t)$ also satisfies the initial condition $u(x, 0) = \sin 3x$, find $u(x, t)$.

10. a) If $u(x, t) = \cos(x - 3t) + 2(x - 3t)^7$, show that $3u_x + u_t = 0$.

b) If $f(s)$ is any smooth function of $s$ and $u(x, t) = f(x - 3t)$, show that $3u_x + u_t = 0$.

11. A function $u(x, y)$ satisfies $3u_x + u_t = 0$. Find an invertible linear change of variables

$$r = ax + bt$$

$$s = cx + dt,$$
where \( a, b, c, d \) are constants, so that in the new \((r, s)\) variables \( u \) satisfies \( \frac{\partial u}{\partial s} = 0. \)

[Remark: There are many possible such changes of variable. The point is to reduce \( 3u_x + u_t = 0 \) to the much simpler \( u_s = 0. \)]

12. Let \( S \) and \( T \) be linear spaces, such as \( \mathbb{R}^3 \) and \( \mathbb{R}^7 \) and \( L : S \to T \) be a linear map; thus, for any vectors \( X, Y \) in \( S \) and any scalar \( c \)

\[
L(X + Y) = LX + LY \quad \text{and} \quad L(cX) = cL(x).
\]

Say \( V_1 \) and \( V_2 \) are (distinct!) solutions of the equation \( LX = Y_1 \) while \( W \) is a solution of \( LX = Y_2 \). Answer the following in terms of \( V_1, V_2, \) and \( W \).

a) Find some solution of \( LX = 2Y_1 - 7Y_2. \)
b) Find another solution (other than \( W \)) of \( LX = Y_2. \)

13. The following is a table of inner ("dot") products of vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w}. \)

<table>
<thead>
<tr>
<th></th>
<th>( \mathbf{u} )</th>
<th>( \mathbf{v} )</th>
<th>( \mathbf{w} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{u} )</td>
<td>4</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>( \mathbf{v} )</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( \mathbf{w} )</td>
<td>8</td>
<td>3</td>
<td>50</td>
</tr>
</tbody>
</table>

For example, \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = 3. \)

a) Find a unit vector in the same direction as \( \mathbf{u}. \)
b) Compute \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}). \)
c) Compute \( \|\mathbf{v} + \mathbf{w}\|. \)
d) Find the orthogonal projection of \( \mathbf{w} \) into the plane \( E \) spanned by \( \mathbf{u} \) and \( \mathbf{v}. \) [Express your solution as linear combinations of \( \mathbf{u} \) and \( \mathbf{v}. \)]
e) Find a unit vector orthogonal to the plane \( E. \)
f) Find an orthonormal basis of the three dimensional space spanned by \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w}. \)

14. Let \( z \) and \( w \) be complex numbers.

a) Write the complex number \( z = \frac{1}{3 + 4i} \) in the form \( z = a + ib \) where \( a \) and \( b \) are real numbers.
b) Show that \( (zw) = \bar{z} \bar{w}. \)
c) Show that \( |z|^2 = z \bar{z}. \)
d) show that \( |zw| = |z||w|. \)
15. If \( z = x + iy \) is a complex number, one way to define \( e^z \) is by the power series

\[
e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^k}{k!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]

(2)

a) Using the usual (real) power series for \( \cos y \) and \( \sin y \), show that

\[e^{iy} = \cos y + i \sin y.\]

b) Use this to show that \( \cos y = \frac{e^{iy} + e^{-iy}}{2} \) and \( \sin y = \frac{e^{iy} - e^{-iy}}{2i}. \)

c) Using equation (2), one can show that \( e^{z+w} = e^z e^w \) for any complex numbers \( z \) and \( w \) (accept this for now). Consequently

\[e^{i(x+y)} = e^{ix} e^{iy}.\]

Use the result of part (a) to show that this implies the usual formulas for \( \cos(x+y) \) and \( \sin(x+y) \).

16. Let \( D \subset \mathbb{R}^2 \) be a bounded (connected) region with smooth boundary \( B \). If \( u(x, y) \) is a “smooth” function, write \( \Delta u = u_{xx} + u_{yy} \) (we call \( \Delta \) the Laplace operator).

Suggestion: First do this problem for a function of one variable, \( u(x) \), so \( \Delta u = u'' \) and, say, \( D \) is the interval \( \{0 < x < 1\} \).

a) Show that \( u\Delta u = \nabla \cdot (u\nabla u) - |\nabla u|^2. \)

b) If \( u(x, y) = 0 \) on \( B \). Show that

\[\iint_D u\Delta u \, dx \, dy = -\iint_D |\nabla u|^2 \, dx \, dy. \]

c) If \( \Delta u = 0 \) in \( D \) and \( u = 0 \) on the boundary \( B \), show that \( u(x, y) = 0 \) throughout \( D \).

[Last revised: January 23, 2015]