Problem Set 1

Due: Thurs. Jan. 22 in class. [Late papers will be accepted until 1:00 PM Friday.]

This is rust remover. It is essentially Homework Set 0 with a few modifications. Notation:
\[ u_t = \frac{\partial u}{\partial t}. \]

This week. Please read all of Chapter 1 in the Haberman text.

1. Let \( u(t) \) be the solution of \( u' = 3u \) with initial value \( u(0) = A > 0 \). At what time \( T \) is \( u(T) = 2A? \)

2. Let \( u(t) \) be the amount of a radioactive element at time \( t \) and say initially, \( u(0) = A > 0 \). The rate of decay is proportional to the amount present, so
\[
\frac{du}{dt} = -cu,
\]
where the constant \( c > 0 \) determines the decay rate. The half-life \( T \) is the amount of time for half of the element to decay, so \( u(T) = \frac{1}{2}u(0) \). Find \( c \) in terms of \( T \) and obtain a formula for \( u(t) \) in terms of \( T \).

3. Let \( \int_0^x f(t) \, dt = e^{\cos(3x)} + A \), where \( f \) is some continuous function. Find \( f \) and the constant \( A \).

4. a) If \( u'' + 4u = 0 \) with initial conditions \( u(0) = 1 \) and \( u'(0) = -2 \), compute \( u(t) \).
   b) Find a particular solution of the inhomogeneous equation \( u'' + 4u = 8 \).
   c) Find a particular solution of the inhomogeneous equation \( u'' + 4u = -4t \).
   d) Find a particular solution of the inhomogeneous equation \( u'' + 4u = -8 - 8t \).
   e) Find the most general solution of the inhomogeneous equation \( u'' + 4u = 8 - 8t \).
   f) If \( f(t) \) is any continuous function, use the method “variation of parameters” (look it up if you don’t know it) to find a formula for a particular solution of \( u'' + 4u = f(t) \).

5. Let \( u(t) \) be any solution of \( u'' + 2bu' + 4u = 0 \). If \( b > 0 \) is a constant, show that \( \lim_{t \to \infty} u(t) = 0 \).

6. a) If \( u'' - 4u = 0 \) with initial conditions \( u(0) = 1 \) and \( u'(0) = -2 \), compute \( u(t) \).
   b) Find a particular solution of the inhomogeneous equation \( u'' - 4u = 8 \).
   c) Find a particular solution of the inhomogeneous equation \( u'' - 4u = -4t \).
7. Say \( w(t) \) satisfies the differential equation

\[
aw''(t) + bw' + cw(t) = 0, \tag{1}
\]

where \( a \) and \( c \) are positive constants and \( b \geq 0 \). Let \( E(t) = \frac{1}{2}[aw'^2 + cw^2] \).

a) Without solving the differential equation, show that \( E'(t) \leq 0 \).

b) Use this to show that \( w(0) = 0 \) and \( w'(0) = 0 \), then \( w(t) = 0 \) for all \( t \geq 0 \).

c) [Uniqueness] Say the functions \( u(t) \) and \( v(t) \) both satisfy the same equation \( \text{(1)} \) and also \( u(0) = v(0) \) and \( u'(0) = v'(0) \). Show that \( u(t) = v(t) \) for all \( t \geq 0 \).

8. Say \( u(x, t) \) has the property that \( \frac{\partial u}{\partial t} = 2 \) for all points \( (x, t) \in \mathbb{R}^2 \).

a) Find some function \( u(x, t) \) with this property.

b) Find the most general such function \( u(x, t) \).

c) If \( u(x, 0) = \sin 3x \), find \( u(x, t) \).

d) If instead \( u \) satisfies \( \frac{\partial u}{\partial t} = 2xt \), still with \( u(x, 0) = \sin 3x \), find \( u(x, t) \).

9. Say \( u(x, t) \) has the property that \( \frac{\partial u}{\partial t} = 3u \) for all points \( (x, t) \in \mathbb{R}^2 \).

a) Find some such function – other than the trivial \( u(x, t) \equiv 0 \).

b) Find the most general such function.

c) If \( u(x, t) \) also satisfies the initial condition \( u(x, 0) = \sin 3x \), find \( u(x, t) \).

10. a) If \( u(x, t) = \cos(x - 3t) + 2(x - 3t)^7 \), show that \( 3u_x + u_t = 0 \).

b) If \( f(s) \) is any smooth function of \( s \) and \( u(x, t) = f(x - 3t) \), show that \( 3u_x + u_t = 0 \).

11. A function \( u(x, y) \) satisfies \( 3u_x + u_t = f(x, t) \), where \( f \) is some specified function.

a) Find an invertible linear change of variables

\[
\begin{align*}
    r &= ax + bt \\
    s &= cx + dt,
\end{align*}
\]
where $a$, $b$, $c$, $d$ are constants, so that in the new $(r, s)$ variables $u$ satisfies $\frac{\partial u}{\partial s} = g(r, s)$, where $g$ is related to $f$ by the change of variables. [Remark: There are many possible such changes of variable. The point is to reduce the differential operator $3u_x + u_t$ to the much simpler $u_s$.]

b) Use this procedure to solve

$$3u_x + u_t = 1 + x + 2t \quad \text{with} \quad u(x, 0) = e^x.$$ 

12. Let $S$ and $T$ be linear spaces, such as $\mathbb{R}^3$ and $\mathbb{R}^7$ and $L : S \to T$ be a linear map; thus, for any vectors $X$, $Y$ in $S$ and any scalar $c$

$$L(X + Y) = LX + LY \quad \text{and} \quad L(cX) = cL(x).$$

Say $V_1$ and $V_2$ are (distinct!) solutions of the equation $LX = Y_1$ while $W$ is a solution of $LX = Y_2$. Answer the following in terms of $V_1$, $V_2$, and $W$.

a) Find some solution of $LX = 2Y_1 - 7Y_2$.

b) Find another solution (other than $W$) of $LX = Y_2$.

13. The following is a table of inner (“dot”) products of vectors $u$, $v$, and $w$.

<table>
<thead>
<tr>
<th></th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>4</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$v$</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$w$</td>
<td>8</td>
<td>3</td>
<td>50</td>
</tr>
</tbody>
</table>

For example, $v \cdot w = w \cdot v = 3$.

a) Find a unit vector in the same direction as $u$.

b) Compute $u \cdot (v + w)$.

c) Compute $\|v + w\|$.

d) Find the orthogonal projection of $w$ into the plane $E$ spanned by $u$ and $v$. [Express your solution as linear combinations of $u$ and $v$.]

e) Find a unit vector orthogonal to the plane $E$.

f) Find an orthonormal basis of the three dimensional space spanned by $u$, $v$, and $w$.

14. Let $z$ and $w$ be complex numbers.

a) Write the complex number $z = \frac{1}{3 + 4i}$ in the form $z = a + ib$ where $a$ and $b$ are real numbers.
b) Show that \( (zw) = \bar{z}\bar{w} \).

c) Show that \( |z|^2 = z\bar{z} \).

d) show that \( |zw| = |z||w| \).

15. If \( z = x + iy \) is a complex number, one way to define \( e^z \) is by the power series

\[
e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]  

(2)
a) Using the usual (real) power series for \( \cos y \) and \( \sin y \), show that

\[
e^{iy} = \cos y + i \sin y.
\]

b) Use this to show that \( \cos y = \frac{e^{iy} + e^{-iy}}{2} \) and \( \sin y = \frac{e^{iy} - e^{-iy}}{2i} \).

c) Using equation (2), one can show that \( e^{z+w} = e^z e^w \) for any complex numbers \( z \) and \( w \) (accept this for now). Consequently

\[
e^{i(x+y)} = e^{ix} e^{iy}.
\]

Use the result of part (a) to show that this implies the usual formulas for \( \cos(x+y) \) and \( \sin(x+y) \).

16. Let \( \mathcal{D} \subset \mathbb{R}^2 \) be a bounded (connected) region with smooth boundary \( \mathcal{B} \). If \( u(x,y) \) is a “smooth” function, write \( \Delta u = u_{xx} + u_{yy} \) (we call \( \Delta \) the Laplace operator). Some people write \( \Delta u = \nabla^2 u \).

**Suggestion:** First do this problem for a function of one variable, \( u(x) \), so \( \Delta u = u'' \) and, say, \( \mathcal{D} \) is the interval \( \{0 < x < 1\} \).

a) Show that \( u\Delta u = \nabla \cdot (u\nabla u) - |\nabla u|^2 \).

b) If \( u(x, y) = 0 \) on \( \mathcal{B} \). Show that

\[
\iint_{\mathcal{D}} u\Delta u \, dx \, dy = -\iint_{\mathcal{D}} |\nabla u|^2 \, dx \, dy.
\]

c) If \( \Delta u = 0 \) in \( \mathcal{D} \) and \( u = 0 \) on the boundary \( \mathcal{B} \), show that \( u(x,y) = 0 \) throughout \( \mathcal{D} \).

17. The temperature \( u(x,t) \) of a certain thin rod, \( 0 \leq x \leq L \) satisfies the heat equation

\[
u_t = u_{xx}
\]
Assume the initial temperature $u(x, 0) = 0$ and that both ends of the rod are kept at a temperature of 0, so $u(0, t) = u(L, t) = 0$ for all $t \geq 0$. What do you anticipate the temperature in the rod will be at any later time $t$?

I hope you suspect that $u(x, t) = 0$ for all $t \geq 0$. Use the following to prove this. Let

$$H(t) = \int_0^L u^2(x, t) \, dx.$$  

a) Show that since the temperature on the ends of the rod is always zero, then $dH/dt \leq 0$ (an integration by parts will be needed). Thus, for any $t \geq 0$ we know that $H(t) \leq H(0)$

b) Since the initial temperature is zero, what is $H(0)$? Why does this imply that $H(t) = 0$ for all $t \geq 0$? Why does this imply that $u(x, t) = 0$ for all points on the rod and all $t \geq 0$?

**Bonus Problem**

[Please give this directly to Professor Kazdan]

B-1 [Generalization of Problem 17 to more space dimensions]. Say a function $u(x, y, t)$ satisfies the heat equation in a bounded region $\Omega \in \mathbb{R}^2$

$$u_t = u_{xx} + u_{yy}$$

and that $u(x, y, t) = 0$ for all points $(x, y)$ on the boundary, $\mathcal{B}$ of $\Omega$ and all $t \geq 0$. Similar to Problem 17, define

$$H(t) = \iint_{\Omega} u^2(x, y, t) \, dx \, dy.$$  

a) Use $u(x, y, t) = 0$ for all points $(x, y)$ on the boundary $\mathcal{B}$ and all $t \geq 0$ to show that $dH/dt \leq 0$ for all $t \geq 0$. [Suggestion: See Problem 16.]

b) If in addition use that the initial temperature is zero, $u(x, y, 0) = 0$ for all points $(x, y) \in \Omega$, to show that $u(x, y, t) = 0$ for all $(x, y) \in \Omega$ and all $t \geq 0$.

[Last revised: January 23, 2015]