Problem Set 3 #2.5.9b

NOTE: In working Homework Set 3 #11 = page 83: 2.5.9b in office hours, it became clear that it requires facts covered in Math 114 and 240 that few people seemed to remember. Here is a brief summary.

Review 1. Say you have a square matrix and want to solve the homogeneous equation $AX = 0$. Clearly a “trivial” solution is $X = 0$. FACT: There is a non-trivial solution $X \neq 0$ if and only if $\det A = 0$. This is particularly simple to use if $A$ is a $2 \times 2$ matrix. For larger matrices the determinant is often unpleasant – or essentially impossible – to compute.

Review 2. If $z = \alpha + i\beta$ is a complex number and $r > 0$, how do you compute $r^z$? Since
\[ e^{\alpha+i\beta} = e^\alpha e^{i\beta} = e^\alpha (\cos \beta + i \sin \beta), \]
we can compute $r^z$ for the special case when $r = e$. For the general case we use the definition
\[ r^z = e^{z \ln r} = e^{\alpha \ln r + i \beta \ln r} = r^\alpha [\cos(\beta \ln r) + i \sin(\beta \ln r)] \]

Review 3: Euler’s Differential Equation The homogenous equation is
\[ Ly =: ax^2 y'' + bxy' + cy = 0. \]
To solve this we try $y = x^q$, where $q$ might be a complex number (see REVIEW 2 above. Then by a straightforward computation,
\[ L(x^q) = ax^2 q(q - 1)x^{q-2} + bxqx^{q-1} + cx^q = [aq(q - 1) + bq + c]x^\alpha \]
Thus, if $q$ is a root of the quadratic polynomial $P(q) = aq(q - 1) + bq + c$, then $L(x^q) = 0$ so $y = x^q$ is a solution of the homogeneous equation $Ly = 0$. Usually, the quadratic polynomial $P(q) = 0$ has two distinct roots, say $q_1$ and $q_2$ (which might be complex!). In that case the general solution of the homogeneous equation $Ly = 0$ is
\[ y = Ax^{q_1} + Bx^{q_2} \]
where $A$ and $B$ are constants. If the coefficients $a$, $b$, and $c$ are real, then complex roots of $P(q) = 0$ occur in conjugate pairs, $q = \alpha \pm i\beta$. Then by equation (2) the general real form of the solution of $Ly = 0$ is
\[ y = Ar^\alpha \cos(\beta \ln x) + Br^\alpha \sin(\beta \ln x) \]
If \( q \) happens to be a double (that is, repeated) root of \( P(q) = 0 \), then the general solution of \( Ly = 0 \) is

\[
y = Ax^q + Bx^q \ln x.
\]

**Problem 2.5.9b** Solve the Laplace equation in the circular sector (polar coordinates!)

\( 0 < a < r < b, \ 0 < \theta < \pi/2 \) with the boundary conditions:

\[
u(r, 0) = 0, \quad u(r, \pi/2) = f(r), \quad u(a, \theta) = 0, \quad u(b, \theta) = 0
\]

Note that in polar coordinates the Laplace equation is

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
\]

Using separation of variables we seek a solution of the form \( u(r, \theta) = \phi(\theta)G(r) \). A routine computation then gives

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dG}{dr} \right) - \frac{1}{\phi} \frac{d^2 \phi}{d\theta^2} = \text{const} = \gamma.
\]

The boundary conditions give

\[
0 = u(r, 0) = \phi(0)G(r), \quad f(r) = u(r, \pi/2) = \phi(\pi/2)G(r)
\]

\[
0 = u(a, \theta) = \phi(\theta)G(a), \quad 0 = u(b, \theta) = \phi(\theta)G(b).
\]

[Last revised: February 3, 2015]