Problem Set 4

Due: Thurs. Feb. 12 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. Please read all of Chapter 4 Sec. 4 and Chapter 3 Sec. 3.1 - 3.3 in the Haberman text.

Note: Exam 1 Tues, Feb. 17, 10:30-11:50. Closed book, no calculators, no cell phones, but you may use one 3 × 5 card with notes on both sides.

Remark: One goal of the Bonus Problem 1 below is to show that if \( \phi(t) \) satisfies

\[
\phi'' + \gamma \phi = 0,
\]

where \( \gamma \) is a const and if \( \phi \) satisfies the periodic boundary conditions

\[
\phi(P) = \phi(0) \quad \text{and} \quad \phi'(P) = \phi'(0),
\]

then \( \phi(t) \) is periodic with period \( P \), that is, \( \phi(t + P) = \phi(t) \) for all \( t \). This justifies why (1) are called “periodic boundary conditions.”

1. p. 83 #2.5.10
2. p. 84 #2.5.13
3. p. 84 #2.5.14
4. Solve the Laplace equation in the annular region \( 1 < r < 2 \) in the plane with boundary conditions (polar coordinates) \( u(1, \theta) = 3 \) and \( u(2, \theta) = 5 \).
5. [Solid Mean Value Property] Let \( u \) satisfy the Laplace equation \( \Delta u = 0 \) in a region \( \Omega \) in the plane \( \mathbb{R}^2 \). The book (p. 79) shows that the for any disk \( D \), the value at the origin is the average of its values on the bounding circle \( C \) of that disk.

Multiplying this formula by \( r \) and then integrating, obtain the “Solid Mean Value Property” for a disk \( D(a) \) of radius \( a \)

\[
u(0, \theta) = \frac{1}{\pi a^2} \int_0^a \left( \int_0^{2\pi} u(r, \theta) \, d\theta \right) r \, dr = \frac{1}{\pi a^2} \int_D u(r, \theta) \, dA,
\]

where \( dA = r \, dr \, d\theta \) is the element of area in polar coordinates.

6. p. 143 # 4.4.6
7. Use separation of variables to solve the wave equation \( u_{tt} = u_{xx} \) (so we are taking \( c = 1 \) for a vibrating string \( 0 < x < \pi \) with initial conditions

\[
    u(x, 0) = 3 \sin 7x + 2 \sin 19x, \quad u_t(x, 0) = 8 \sin 5x,
\]

and boundary conditions (fixed end points) \( u(0, t) = 0, \ u(\pi, t) = 0 \).

8. a) Let \( u(x, t) \) be a solution of the wave equation \( u_{tt} = u_{xx} \) for a vibrating string \( 0 < x < L \) whose end points are fixed: \( u(0, t) = 0, \ u(L, t) = 0 \) and define the “Energy”, \( E(t) \), by

\[
    E(t) = \frac{1}{2} \int_0^L (u_t^2 + u_x^2) \, dx.
\]

Show that energy is conserved, \( dE/dt = 0 \). [Suggestion: Integrate by parts. Since \( u = 0 \) on the boundary, then also \( u_t = 0 \) there.]

b) If \( u(x, 0) = 0 \) and \( u_t(x, 0) = 0 \), what can you conclude?

c) Generalize this to the motion \( u(x, y, t) \) of a vibrating membrane, \( \Omega \subset \mathbb{R}^2 \) so \( u_{tt} = \Delta u \) in \( \Omega \) whose boundary, \( \partial \Omega \), is fixed: \( u(x, y, t) = 0 \) for all \( (x, y) \) on \( \partial \Omega \) and all \( t \geq 0 \). Here

\[
    E(t) = \frac{1}{2} \int_D (u_t^2 + |\nabla u|^2) \, dx \, dy.
\]

Show that energy is conserved.

**Bonus Problem**

[Please give this directly to Professor Kazdan]

B-1 Say a function \( u(t) \) satisfies the differential equation

\[
    u'' + b(t)u' + c(t)u = 0
\]

on the interval \( [0, A] \) and that the coefficients \( b(t) \) and \( c(t) \) are both bounded, say \( |b(t)| \leq M \) and \( |c(t)| \leq M \) (if the coefficients are continuous, this is always true for some \( M \)).

a) Define \( E(t) := \frac{1}{2}(u'^2 + u^2) \). Show that for some constant \( \gamma \) (depending on \( M \)) we have \( E'(t) \leq \gamma E(t) \). [Suggestion: use the simple inequality \( 2xy \leq x^2 + y^2 \).]

b) Show that \( E(t) \leq e^{\gamma t} E(0) \) for all \( t \in [0, A] \). [Hint: First use the previous part to show that \( (e^{-\gamma t} E(t))' \leq 0 \).

c) In particular, if \( u(0) = 0 \) and \( u'(0) = 0 \), show that \( E(t) = 0 \) and hence \( u(t) = 0 \) for all \( t \in [0, A] \). In other words, if \( u'' + b(t)u' + c(t)u = 0 \) on the interval \( [0, A] \) and that the functions \( b(t) \) and \( c(t) \) are both bounded, and if \( u(0) = 0 \) and \( u'(0) = 0 \), then the only possibility is that \( u(t) \equiv 0 \) for all \( t \geq 0 \).
d) Use this to prove the uniqueness theorem: if \( v(t) \) and \( w(t) \) both satisfy equation

\[
u'' + b(t)u' + c(t)u = f(t)
\]  \hspace{1cm} (3)

and have the same initial conditions, \( v(0) = w(0) \) and \( v'(0) = w'(0) \), then \( v(t) \equiv w(t) \) in the interval \([0, A]\).

e) Assume the coefficients \( b(t) \), \( c(t) \), and \( f(t) \) in equation (3) are periodic with period \( P \), that is, \( b(t + P) = b(t) \) etc. for all real \( t \). If \( \phi(t) \) is a solution of equation (3) that satisfies the periodic boundary conditions \( (1) \), show that \( \phi(t) \) is periodic with period \( P \).

[Last revised: June 28, 2015]