Problem Set 6

DUE: Thurs. March 5 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. Please read Chapter 5, Sections 5.1 – 5.7 in the Haberman text.

1. Let \( f(x) = x^2, \ 0 \leq x \leq \pi. \)
   a) Extended \( f \) to all real \( x \) as an even function that is \( 2\pi \) periodic. For which values of \( x \) does its Fourier series converge point wise to \( f \)? At the point(s) where it does not converge, to what does it converge?
   b) Extended \( f \) to all real \( x \) as an odd function that is \( 2\pi \) periodic. For which values of \( x \) does its Fourier series converge point wise to \( f \)? At the point(s) where it does not converge, to what does it converge?

2. Let \( f \) be a \( 2\pi \) periodic function whose first derivative is continuous. How are the Fourier coefficients of \( f \) and \( f' \) related? [Suggestion: Integrate by parts.]

3. [Related to p. 129, 3.6.2] Let \( f \) have the complex Fourier series \( f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}. \)
   a) If \( f \) is real-valued, show that \( c_{-n} = \overline{c_n}. \)
   b) If \( c_{-n} = \overline{c_n}, \) show that \( f \) is real-valued.

4. Let \( f(x) \) be a \( 2\pi \) periodic real function with (complex) Fourier series \( f(x) = \sum_{-\infty}^{\infty} a_n e^{inx} \)
   and let \( \alpha \) be a real constant. We seek a \( 2\pi \) periodic solution \( u(x) \) of
   \[ u'' + \alpha u = f(x) \]
   as a Fourier series \( u(x) = \sum_{-\infty}^{\infty} c_n e^{inx}. \)
   a) Formally find the desired coefficients \( c_n \) in terms of the Fourier coefficients \( a_n \) of \( f. \)
   b) Are there any values of \( \alpha \) for which there are difficulties? Explain. In particular, if \( \alpha = k^2 \) for some integer \( k \), show that a necessary condition for a solution to exist is that
   \[ \int_{-\pi}^{\pi} f(x) \cos kx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} f(x) \sin kx \, dx = 0. \]

5. a) Find the \( 2\pi \) periodic Fourier series for \( f(x) = \begin{cases} 0 & \text{for } -\pi \leq x \leq 0 \\ x & \text{for } 0 \leq x < \pi \end{cases}. \)
   b) Use the boxed theorem p. 123 (top) to find the Fourier series for \( f(x) = \begin{cases} 0 & \text{for } -\pi \leq x \leq 0 \\ x^2 & \text{for } 0 \leq x < \pi \end{cases}. \)
Remark: As is pointed out near the bottom of p. 124 (#2), since the Fourier series for this \( f(x) \) has \( a_0 \neq 0 \), its integral involves \( a_0 x \) so you will also need the \( 2\pi \) periodic Fourier (sine) series for \( x \):

\[
x = 2 \left[ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots + (-1)^{n+1} \frac{\sin nx}{n} + \cdots \right].
\]

6. Use d'Alembert's formula (p. 508, eq 11.2.36) to solve the wave equation \( u_{tt} = c^2 u_{xx} \) for \(-\infty < x < \infty\) with the following initial conditions:

a) \( u(x, 0) = \frac{1}{1 + x^2} \), and \( u_t(x, 0) = 0 \).

b) \( u(x, 0) = 0 \) and \( u_t(x, 0) = 4 \).

7. d'Alembert's formula (p. 508, eq 11.2.36) gives a formula for the solution of \( u_{tt} = c^2 u_{xx} \) for \(-\infty < x < \infty\) with \( u(x, 0) = f(x) \) and \( u_t(x, 0) = g(x) \).

The goal of this problem is to show how to modify d'Alembert's formula to solve the initial value problem for the wave equation on the semi-infinite interval \( x \geq 0 \) given \( u(x, 0) = F(x) \) and \( u_t(x, 0) = G(x) \), where \( F \) and \( G \) are only defined for \( x \geq 0 \) but you also have the additional boundary condition \( u(0, t) = 0 \) for all \( t \geq 0 \). You may assume the compatibility conditions \( F(0) = 0 \) and \( G(0) = 0 \).

a) In the d'Alembert formula, if both \( f(x) \) and \( g(x) \) are odd smooth functions, directly from this formula show that the solution, \( u(x, t) \) is an odd function of \( x \).

Consequence: \( u(0, t) = 0 \) for all \( t \geq 0 \)

b) For the semi-infinite interval problem where \( x \geq 0 \), \( F(x) \) and \( G(x) \) are defined only for \( x > 0 \). Let \( f_{odd} \) and \( g_{odd} \) be their extensions to all \( x \in \mathbb{R} \) as odd functions. Then let \( u(x, t) \) be the solution that d'Alembert’s formula gives for the solution of the wave equation for all real \( x \).

Show that if you only use this \( u(x, t) \) for \( x \geq 0 \), it is a solution of the wave equation for \( x > 0 \) with \( u(x, 0) = F(x), u_t(x, 0) = G(x) \) with the end at \( x = 0 \) fixed: \( u(0, t) = 0 \) for all \( t > 0 \). [For \( u \) to be smooth enough, assume that \( F \) and \( G \) and their odd extensions are smooth.]

8. Say \( u_{tt} = 9u_{xx} \) for all \(-\infty < x < \infty \) with \( u(x, 0) = f(x) \) and \( u_t(x, 0) = g(x) \). Find the largest interval \( J = \{a \leq x \leq b\} \) where modifying \( f \) and/or \( g \) inside this interval can change the value of \( u(6, 5) \).

[Last revised: March 4, 2015]